Petri Nets Step Transitions and Proofs in Partially Commutative Linear Logic

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Abstract: We encode the execution of Petri nets in Partially Commutative Linear Logic, an intuitionistic logic introduced by Ph. de Groote which contains both commutative and non commutative connectives. We are thus able to faithfully represent the concurrent firing of Petri nets as long as it can be depicted by a series-parallel order. This coding is inspired from the description of context-free languages by Lambek grammars.

This report is an extended version (with complete proofs) of an article to appear in the proceedings of the Logic Colloquium 1999 (Utrecht).

Key-words: Linear logic; Petri nets; Categorial Grammars; AMS: 03B47, 03B60, 03B70, 03F05, 03F52, 68Q85

(Résumé : tsvp)

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Exécution non séquentielle des réseaux de Petri et preuves en logique linéaire partiellement commutative

Résumé : Nous décrivons l'exécution d'un réseau de Petri dans la logique linéaire partiellement commutative, une logique intuitionniste introduite par Ph. de Groote qui contient et des connecteurs commutatifs et des connecteurs non commutatifs. Nous sommes ainsi capable de décrire fidèlement l'exécution en parallèle d'un réseau de Petri, du moins tant que celle-ci reste un ordre série-parallèle. Ce codage s'inspire de la description des langages algébriques par les grammaires de Lambek.

Ce rapport est la version complète (incluant toutes les démonstrations) d'un article à paraître dans les actes du Logic Colloquium 1999 (Utrecht).

Mots-clé : Logique linéaire; Réseaux de Petri; Grammaires Catégorielles; AMS: 03B47, 03B60, 03B70, 03F05, 03F52, 68Q85

Contents

This report is the complete version of an article [25]; it is similar but it includes some extra material which is absent from the published version:

- *the proof of cut-elimination (in the appendix B),*
- *the counter example by Künzle to a previous attempt to logically describe the maximally concurrent firings of a Petri net (in the appendix C)*
- *the proofs of several propositions.*

1 Presentation

Since the early days of linear logic, various representations of Petri nets have been proposed — [28, 8, 12] include good surveys; nevertheless these codings are not fully satisfactory. On the logical side the use of proper axioms is not pleasant, and to avoid it the only solution is to include modalities while it is clearly a multiplicative phenomenon; in particular as Petri net accessibility is decidable, it should be encoded in a decidable fragment of linear logic, and it is not yet known whether linear logic with modalities is decidable. On the concurrency side the absence of any record of the execution in the sequent which is proved leaves out a number of interesting questions, like Petri net synthesis or the search of efficient executions. For instance, even the sophisticated treatment of Gehlot [9] only takes into account structural parallelism, and fails to find an efficient execution due to the presence of the marking as shown by Künzle [15, 22].

Here we propose a rather different representation which focuses on events and executions. This is made possible by using the partially commutative calculus, here denoted by PCLL, introduced by Philippe de Groote in 96 [14] — an extension of the published version, to be precise. In this intuitionistic calculus one both have non commutative connectives of the Lambek calculus [16, 24] and the usual commutative connectives of multiplicative linear logic [11]. This kind of calculus has then been extended to a classical setting by Paul Ruet [26], and further studied by Michele Abrusci and Paul Ruet [1], Akim Demaille [7].

Roughly speaking it is possible to combine the commutative and non commutative logical connectives by handling structured contexts. These contexts are series-parallel partial orders, i.e. are generated by two kinds of commas: the commutative comma, denoted by $\{\dots,\dots\}$ corresponding to the disjoint sum of two contexts, introducing no order between its components, and the non commutative comma $(\dots;\dots)$ introducing order between its components: every formula in the first component is before every formula in its second component. This structure allows to take into account the relationship between the commutative and non commutative products, $\otimes$ and $\odot$ respectively corresponding to context operations $\{\dots,\dots\}$ and $(\dots;\dots)$. This relationship is simply the inclusion of series-parallel orders axiomatized in [3]. It should be observed that this relation is more general than the possibility to replace a commutative product by a non commutative one. Indeed, inclusion of series-parallel orders is basically the distributive law of concurrency viewed as a reduction and not as an equality: $\{(a;b),(a';b')\} \to (\{a,a'\};\{b,b'\})$ that is if one has to perform a before b and a' before b' without any further constraint, one can in particular perform a and a' simultaneously and perform thereafter b and b' simultaneously. This law cannot be derived by replacing commutative product(s) with non commutative product(s). From a logical perspective it is worth noticing that this relationship is only possible in an intuitionistic calculus as shown in [7, Chapter 6]. For instance the classical calculus considered by [26, 1] only allows for the replacement of commutative products with non commutative products but not for the distributive law. It is also worth noticing that the logical system allows for either relationship: a weaker order entails a stronger order, or the converse; of course for a concurrency interpretation only the first system is relevant.

As usual a marking with $n(P)$ tokens in the place P will be denoted by the formula $\bigotimes_P P^{n(P)}$ where $\otimes$ is the *commutative product* and P^k a short hand for $P \otimes \cdots \otimes P$ k-times. An event e consuming the marking $\mathsf{Pre}[e]$ and producing the marking $\mathsf{Post}[e]$ will be denoted by a formula $\mathsf{Pre}[e] \setminus \mathsf{Post}[e]$ with $\setminus$ being a non commutative implication: so it is not that far from the usual translation $\mathsf{Pre}[e] \multimap \mathsf{Post}[e]$, except that one has $(\mathsf{Pre}[e]; \mathsf{Pre}[e] \setminus \mathsf{Post}[e]) \vdash \mathsf{Post}[e]$, but not $(\mathsf{Pre}[e] \setminus \mathsf{Post}[e]; \mathsf{Pre}[e]) \vdash \mathsf{Post}[e]$.

We are given a partially ordered multiset of events ϕ, and we can fire simultaneously any subset of minimal events, until all events in ϕ are fired. We will prove that such an execution is possible from an initial marking M and yields the end marking N if and only if the calculus PCLL proves $(M;\phi) \vdash N$.

So in fact we are turning a universal statement into an existential one: *every sequence of step transitions of ϕ is possible from M and yields to N* is shown to be equivalent with *there exists a proof of* $(M;\phi) \vdash N$. To be a bit more precise, our coding and the proof of its correctness handle *executive* sequents $\Gamma \vdash C$ that are the meaningful sequents:

- C is a marking
- formulae of Γ are either markings ($\otimes$-only formulae) or events ($M \setminus N$ with M and N markings)
- the formulae of Γ are endowed with a series-parallel partial order

The provability of an executive sequent means that the concurrent execution of the events in Γ (endowed with the order induced by Γ) leads from the sum of the markings in Γ to the marking C.

Provability corresponds to the possibility of executing the corresponding step transitions, but what do proofs correspond to? They allow to trace every produced or consumed token, which are distinguished in a given proof. Of course the ideal representation of proofs would be proof nets, which exactly identify all proofs depicting the same consumption/production, but up to now proof nets for this calculus do not exist (there is not a yet a sound and complete correctness criterion for recognizing proofs from incorrect proof structures).

But as far as provability is concerned, tokens are not distinguished. Therefore complete models as in [7] may be used instead of proofs to observe the behaviour of Petri nets, since making a distinction between tokens is an artifact — that is nevertheless useful as in [13, 4].

The kind of Petri net execution that we take into account is step transition, where the steps are lower-closed subsets of the partial order of events which expresses the causal constraints in the execution. Step transition are studied in [20] but they are just multisets of events: they are not assumed to be the lower closed subsets of an order on events. On the other hand, the occurrence nets of [13, 4] unfold the causality of a Petri net into a partial order (acyclic graph) where vertices are alternatively events and places. The partial orders of events we consider here are sub-orders of this general order, restricted to events. So our approach is mixed: we consider step transitions, but these steps are lower closed subsets of this general partial orders. We only can deal with series-parallel orders, and in this case we are able to replace a complicated statement into the existence of a proof in a multiplicative system. At first sight this work also share some ingredients with the algebraic approach of [18] which describe Petri nets by monoidal categories, for instance the distributive law which for us is a reduction and for them an equality, or transitions between markings. This not so surprising, as models of multiplicative calculi are monoidal categories, but in fact the connection, if any, is far from obvious: we superimpose a commutative and a non commutative structure, our objects are not only markings but all executions, and our morphisms are proofs expressing that if an execution is possible, so is another, while their morphisms are executions leading from a marking to a marking.

We first recall basic definition and results regarding step transition for Petri nets, using the logical notation. The essential result we need is a substitution property for step transition: although not difficult, it is bit tedious, so the proof is postponed to an appendix. Next we introduce the PCLL calculus, and the properties needed for our coding. After a small example we establish the faithfulness of the encoding in both directions. We end with future prospects: indeed our logical description naturally leads to high order Petri nets (mobile nets, where events can produce and consume events, etc.), to Petri nets with credits (where one can assume that some tokens are present provided they are consumed afterwards) and also suggest a new approach to ordinary Petri net synthesis because of the connection between formal grammar and Lambek calculus.

Set theoretic notation

$A - B = \{x \in A | x \notin B\}$ is only used when $A \supseteq B$. $A \uplus B$ denotes either multiset union or the union of two disjoint sets.

Acknowledgments

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2 Petri nets and their concurrent firing

2.1 Petri nets: definition and notation

A Petri net is defined by a set $\mathcal{P}$ of *places* and a set $\mathcal{E}^{\otimes\backslash\otimes}$ of *events*. In our view the initial marking of a Petri net is not part of its definition.

2.1.1 Places and markings

Usually, a marking is defined as a function from $\mathcal{P}$ to $\mathbf{N}$ which expresses how many tokens there are in each place. In order to get as close as possible to our coding, let us define a marking as an element of the free commutative monoid over the places: it is clearly an equivalent definition. Composition will be denoted by $\otimes$, and the unit by $\mathbf{1}$ (the empty marking). As usual, A^n with $n \in \mathbf{N}$ and $A \in \mathcal{P}$ is defined by: $A^0 = \mathbf{1}$ and $A^{n+1} = A \otimes A^n$. Due to the equations holding in the free commutative monoid over $\mathcal{P}$, there are many expressions denoting the same marking; for instance $\mathbf{1} \otimes ((A \otimes B) \otimes A)$, $A^2 \otimes B$, $A \otimes (A \otimes B)$ and $(B \otimes A) \otimes A$ all stand for the marking containing two tokens in the place A and one in the place B. Let us denote by $\Pi^\otimes$ such marking expressions:

$$\Pi^\otimes ::= \mathcal{P} \mid \mathbf{1} \mid \Pi^\otimes \otimes \Pi^\otimes$$

Given an expression M in $\Pi^\otimes$ and a place A of $\mathcal{P}$ the expression M_A denotes the number of occurrences of A in M, that is the number of tokens of M in A. This number only depends on the marking and not on the expression, so it is definable for a marking.

A marking M' is said to contain a marking M, whenever for each place A, $M'_A \geq M_A$; in this case we write $M' \sqsupseteq M$. If so, then there exists another unique marking X such that $X \otimes M = M'$; this marking X will be denoted by $M' \oslash M$.

2.1.2 Events

Each $e \in \mathcal{E}^{\otimes\backslash\otimes}$ is associated with a behavior, that is a pair of maps $\mathsf{Pre}[e]$ and $\mathsf{Post}[e]$ from places to $\mathbf{N}$. These two maps may be viewed as a pair of markings, indicating how many tokens, for each place A are taken off and put in A by the event e; according to the previous notation, these two numbers will be denoted by

$\mathsf{Pre}[e]_A$ and $\mathsf{Post}[e]_A$. The behavior of an event e will be denoted by $\mathsf{Pre}[e] \setminus \mathsf{Post}[e]$ where expressions of $\Pi^{\otimes}$ are allowed for $\mathsf{Pre}[e]$ and $\mathsf{Post}[e]$.

Let us mention a classical distinction among Petri nets. A Petri net is said to be unlabeled whenever no two events have the same behavior:

$$\forall e_1, e_2 \in \mathcal{E}^{\otimes \setminus \otimes} \ (\mathsf{Pre}[e_1] = \mathsf{Pre}[e_2] \wedge \mathsf{Post}[e_1] = \mathsf{Post}[e_2]) \Rightarrow e_1 = e_2$$

When a behavior makes use of expressions in $\Pi^{\otimes}$ equality is the equality of the underlying markings, for instance: $(A \otimes B) \setminus (\mathbf{1} \otimes A) = (B \otimes A) \setminus A$. As it stands, this work does not apply to labeled Petri nets, and this allows to identify an event with its behavior. [1]

Another classical property of Petri nets is purity: for each event e_i and for each place A one has $\mathsf{Pre}[e_i]_A.\mathsf{Post}[e_i]_A = 0$; that is an event never puts any token in a place where he takes some. This distinction is not relevant to our study which works no matter whether Petri nets are pure or not.

2.2 Firing a Petri net according to a partially ordered multiset of events

Given a Petri net $\mathbf{R}$ which will remain implicit, we are to define a relation

$$M \xrightarrow{\phi}$$

where

- ϕ is a partially ordered multiset of events of $\mathbf{R}$ (the definition follows)
- M is a marking $\mathbf{R}$

The meaning of this relation is that the Petri net $\mathbf{R}$, provided with the initial marking M allows the execution of the partially ordered multiset of events ϕ. Such an execution according to ϕ consists in firing simultaneously any set of minimal events until all events are fired. Let us define this precisely.

[1] Nevertheless our coding could be adapted by using a lexicon as in a Lambek grammar [16, 24] (events are provided with a finite number of formulae by the lexicon). The case of rigid Lambek grammars (one formula per event), would correspond to labeled Petri nets (there can be two events with the same behavior, but each event has a single behavior) while general Lambek grammars would correspond to the extended case in which an event may have several behaviors.

2.2.1 Partially ordered multisets (pomsets) and substitution

An *ordered enumeration of elements* of E is a triple $(X, <, f)$ where X is a set, $<$ is a partial strict order on X and f a map from X to E. A *partially ordered multiset (pomset)* of elements of E is an equivalence class of ordered enumerations where the equivalence $(X, <, f) \sim (X', <', f')$ is defined by: there exists an order isomorphism h from $(X, <)$ to $(X', <')$ such that $f'(h(x)) = f(x)$ for all $x \in X$. We only consider finite pomsets over E, i.e. X is always finite. When $<$ is a linear order a pomset over E is a finite sequence over E and when $<$ is empty a pomset over E is a multiset over E. We often say "let $(X, <, f)$ be a pomset over E", as our definitions or constructions do not depend on the ordered enumeration which represents the pomset.

Let $(X, <)$ and $(Y, \prec)$ be two partially ordered sets, with $a \in X$ and $X \cap Y = \emptyset$. The order $X[a := Y]$ is the order $\triangleleft$ over $(X - \{a\}) \uplus Y$ defined by:

- $\triangleleft|_{X-\{a\}} =< |_{X-\{a\}}$,
- $\triangleleft|_Y =\prec$,
- $\forall x \in X - \{a\}\ \forall y \in Y \quad x \triangleleft y \Leftrightarrow x < a$
- $\forall x \in X - \{a\}\ \forall y \in Y \quad y \triangleleft x \Leftrightarrow a < x$.

Let $\phi = (X, <, f)$ and $\psi = (Y, \prec, g)$ be two partially ordered enumerations with $a \in X$, $X \cap Y = \emptyset$. The *substitution* $\phi[a := \psi]$ of a by ψ in ϕ is the ordered enumeration $((X - \{a\}) \uplus Y, \triangleleft, h)$ where $\triangleleft$ is defined as above and and where $h(x) = f(x)$ for x in $(X - \{a\})$ and $h(y) = g(y)$ for $y \in Y$.

Given two ordered enumerations $(X, <, f) \sim (X', <', f')$, the order isomorphism being $h : X \tilde{\mapsto} X'$ and two ordered enumerations $(Y, \prec, g) \sim (Y', \prec', g')$ we have $(X, <, f)[a := (Y, \prec, g)] \sim (X', <', f')[h(a) := (Y', \prec', g')]$. Consequently when it is clear which occurrence of $e \in E$ is substituted we will write the abusive notation $\phi[e := \psi]$. [2]

[2]In case there are several twin occurrences of e in ϕ, the corresponding partially ordered multiset does not depend on which occurrence of e has been substituted by ψ. [Two elements x and y of an order are said to be twins whenever they cannot be compared and $x < z \Leftrightarrow y < z$ et $z < x \Leftrightarrow z < y$ for all $z \notin \{x, y\}$.]

2.2.2 Concurrent execution of a partially ordered multiset of events

Let $\phi = (X, <, f)$ be an ordered enumeration representing a partially ordered multiset of events of a Petri net $\mathbf{R}$.

Let Y be a subset of X; let us extend $\mathsf{Pre}[]$ and $\mathsf{Post}[]$ to a set of events in the simplest way: $\mathsf{Pre}[Y] = \otimes_{y \in Y} \mathsf{Pre}[f(y)]$ and $\mathsf{Post}[Y] = \otimes_{y \in Y} \mathsf{Post}[f(y)]$

Definition 1 *The relation $M \xrightarrow{\phi}$ holds whenever for all lower-closed*[3] *subsets Y of $(X, <)$ one has:*

$$(M \otimes \mathsf{Post}[Y]) \oslash \mathsf{Pre}[Y] \sqsupseteq \mathsf{Pre}[\mathcal{F}_\phi(Y)]$$

where $\mathcal{F}_\phi(Y)$ is the frontier *of Y defined by*

$$\mathcal{F}_\phi(Y) = \{x \in X - Y | \forall z \in X \; z < x \Rightarrow z \in Y\}$$

Let us explain the intuitive meaning of this definition. The partial order depicts time constraints on the firing of the occurrences of events in X. As the firing respects the time constraints, any set of events that have been fired is a lower-closed subset of X. The events in $\mathcal{F}_\phi(Y)$ are the minimal elements of the complement of $\overline{Y}$ in X (the order on $\overline{Y}$ being $< |_{\overline{Y}}$), hence the events in the frontier of Y are the ones that can be fired next. Consequently the above definition simply says that whatever possible (i.e. lower-closed) part Y of X has been fired there are enough tokens left to fire simultaneously all the events that come next.

Firstly let us observe that this definition makes sense for pomsets of events:

Proposition 2 *If ϕ and ψ are two partially ordered enumerations describing the same partially ordered multiset of events i.e. $\phi \sim \psi$; then $M \xrightarrow{\phi}$ if and only if $M \xrightarrow{\psi}$: so we can speak of the execution of a partially ordered multiset of events.*

Here are some remarks on this definition:

- The notation $\oslash$ assumes that $(M \otimes \mathsf{Post}[Y]) \sqsupseteq \mathsf{Pre}[Y]$; if this did not hold, then there would exist a smaller Y' for which this expression would be meaningful, such that the $\sqsupseteq$ fails.

[3] Y is said to be a lower-closed subset of X whenever $\forall z \in X \; (\exists y \in Y \; z \leq y) \Rightarrow z \in Y$

- One can also define $M \xrightarrow{\phi}$ by: for each anti-chain U letting $T = \{x | \forall u \in U\ \neg(x \geq u)\}$, one has: $M \otimes \mathsf{Post}[T] \oslash \mathsf{Pre}[T] \sqsupseteq \mathsf{Pre}[U]$

- Self concurrency is allowed: indeed the frontier of a lower-closed subset Y may contain several occurrences of the same event.

Whenever $M \xrightarrow{\phi}$ the end-marking is uniquely determined:

Proposition 3 *If $M \xrightarrow{\phi}$, then there is a unique marking N such that the final marking obtained by firing ϕ from the initial marking M is N. This marking is defined by $N = M \otimes \mathsf{Post}[X] \oslash \mathsf{Pre}[X]$ — the $\oslash$ makes sense: take $Y = X$ in the definition. In this case we write:*

$$M \xrightarrow{\phi} N$$

Proposition 4 *let $\phi = (X, <, f)$ be a partially ordered enumeration of events.*

1. *If $M \xrightarrow{\phi} M'$ and if $\psi = (X, \prec, f)$ is an order extension of ϕ ($\forall x, x' \in X \quad x < x' \Rightarrow x \prec x'$) then $M \xrightarrow{\psi} M'$.*

2. *In particular when $M \xrightarrow{\phi} M'$ holds, so does $M \xrightarrow{\psi} M'$ for each linearization ψ of ϕ .*

Remark 5 *The converse of this latter point fails, as can be observed from the Petri net $\mathbf{R}$ defined by:*

- *One place A*

- *Two events: $a = A \setminus A$ and $b = A \otimes A \setminus A \otimes A$*

Consider the (multi)set with a single occurrence of each event a and b, and let us consider the three possible pomsets of events on this (multi)set of events:

- *$\{a, b\}$ (empty strict order)*

- *$(a; b)$ (first linearization, $a < b$)*

- *$(b; a)$ (second linearization, $b < a$)*

Since $A \otimes A \not\sqsupseteq (A \otimes A) \otimes A$, one does not *have $A \otimes A \xrightarrow{\{a,b\}}$ but both the two possible linearizations of this partially ordered set of events can be fired:*
$A \otimes A \xrightarrow{(a;b)} A \otimes A$ *and* $A \otimes A \xrightarrow{(b;a)} A \otimes A$

Let us define the minimum marking M^ϕ of a partially ordered multiset of events ϕ. Letting $L(\phi)$ be the set of the lower-closed subsets of ϕ it is defined by:

$$M^\phi{}_A = \max_{Y \in L(\phi)} \mathsf{Pre}[\mathcal{F}(Y)]_A - \mathsf{Post}[Y]_A + \mathsf{Pre}[Y]_A$$

Observe that $M^\phi{}_A \geq 0$, because of the case $Y = \emptyset$. The name minimum marking is justified by the following fairly obvious proposition:

Proposition 6 *One has $M^\phi \xrightarrow{\phi}$, and if $M \xrightarrow{\phi}$ then $M \sqsupseteq M^\phi$*

PROOF : For each lower closed subset Y of ϕ and for each place A one has:

$$M^\phi{}_A + \mathsf{Post}[Y]_A - \mathsf{Pre}[Y]_A \geq \mathsf{Pre}[\mathcal{F}(Y)]_A$$

and therefore $M^\phi \otimes \mathsf{Post}[Y] \oslash \mathsf{Pre}[Y] \sqsupseteq \mathsf{Pre}[\mathcal{F}(Y)]$

If $N \not\sqsupseteq M^\phi$ then there exists a place A such that $N_A < M^\phi{}_A$ — as $N_A \geq 0$ this yields $M^\phi{}_A > 0$. Because $M^\phi{}_A > 0$ and by definition of M^ϕ there exists a lower closed subset Y_0 of ϕ such that

$$M^\phi{}_A = \mathsf{Pre}[\mathcal{F}(Y_0)]_A - \mathsf{Post}[Y_0]_A + \mathsf{Pre}[Y_0]_A$$

and thus we would have

$$N_A + \mathsf{Post}[Y_0]_A - \mathsf{Pre}[Y_0]_A < \mathsf{Pre}[\mathcal{F}(Y_0)]_A$$

$\diamond$

2.2.3 Substitution in an execution

Proposition 7 (substitution property) *Let*

- *$\phi = (X, <, f)$ be a partially ordered enumeration of events containing an occurrence x of $P \setminus Q$ (i.e. $f(x) = P \setminus Q$)*

- $\psi = (Y, \prec, g)$ *be a partially ordered enumeration of events, such that* $Y \cap X = \emptyset$ *and such that* $P \xrightarrow{\psi} Q$

then one has:

1.
$$(a) : \left(M_0 \xrightarrow{\phi}\right) \quad \Longrightarrow \quad \left(M_0 \xrightarrow{\phi[x:=\psi]}\right) \ : (b)$$

2. If moreover P *is the minimum marking for* ψ*, then the converse also holds:*
$$(b) : \left(M_0 \xrightarrow{\phi[x:=\psi]}\right) \quad \Longrightarrow \quad \left(M_0 \xrightarrow{\phi}\right) \ : (a)$$

In both cases when (a) *and* (b) *hold, the final markings are equal.*

PROOF : The equality of the final markings is obvious. The proof of the main part is rather tedious and postponed to the appendix A. $\diamond$

Here is a rather obvious proposition which will allow for the logic to mix events and markings; indeed in the logical model, events are constructed out of markings, and this proposition will be needed for events to appear at the right places in the order on events.

Proposition 8 *Let* $\psi[\mathbf{1} \setminus M] = (X, <, f)$ *be a pomset of events containing an occurrence* x *of the event* $\mathbf{1} \setminus M$ *(i.e.* $f(x) = \mathbf{1} \setminus M$*) and let* $\psi[] = (X - \{x\}, <', f')$ *be the partially ordered multiset obtained by suppressing this occurrence of* $\mathbf{1} \setminus M$*, and taking the induced order on the remaining occurrences of events (*$<' =< |_{X-\{x\}}$ *and* $f' = f|_{X-\{x\}}$*).*

If $X \xrightarrow{\psi[\mathbf{1}\setminus M]} Y$ *then* $X \otimes M \xrightarrow{\psi[]} Y$

PROOF : Let us use the alternative definition with antichains. Let U be an antichain of $\phi[]$, and let T be the subset of $\phi[]$ defined by

$$T = \{x \in \phi[] \mid \forall u \in U \ \neg(x \geq u)\}$$

As U also is an antichain of $\phi[\mathbf{1} \setminus M]$ and letting T' be the lower closed subset of $\phi[\mathbf{1} \setminus M]$ defined by

$$T' = \{x \in \phi[\mathbf{1} \setminus M] | \forall u \in U \ \neg(x \geq u)\}$$

we have:

$$X \otimes \mathsf{Post}[T'] \oslash \mathsf{Pre}[T'] \sqsupseteq \mathsf{Pre}[U]$$

Now observe that as a set T' is either T or $T \cup \{\mathbf{1} \setminus M\}$. Indeed if x is not the suppressed occurrence of $1 \setminus M$ then the statement $\forall u \in U\ \neg(x \geq u)$ is either true in $\phi[]$ and in $\phi[\mathbf{1} \setminus M]$ or false in both, because the order $\phi[]$ is the restriction of the order $\phi[\mathbf{1} \setminus M]$. In the first case, we immediately have

$$X \otimes M \sqsupseteq X \otimes \mathsf{Post}[T] \oslash \mathsf{Post}[T] \sqsupseteq \mathsf{Pre}[U]$$

and in the second case, since $\mathsf{Pre}[T] = \mathsf{Pre}[T']$ (because $\mathsf{Pre}[\mathbf{1} \setminus M] = \mathbf{1}$) and $\mathsf{Post}[T] = \mathsf{Post}[T'] \otimes M$, we have:

$$X \otimes M \otimes \mathsf{Post}[T] \oslash \mathsf{Pre}[T] \sqsupseteq \mathsf{Pre}[U]$$

$\diamond$

2.2.4 Series or parallel composition of executions

Although we shall come back more precisely on these notions for the logical calculus (in paragraph 3.2) we need a few properties of these operations on partial orders.

Given two partially ordered enumerations $(X, <, f)$ and $(Y, \prec, g)$ with $X \cap Y = \emptyset$ we define:[4]

- their *parallel-composition* $(X \uplus Y, < \uplus \prec, f \uplus g)$
- their *series-composition* $(X \uplus Y, < \uplus \prec \uplus (X \times Y), f \uplus g)$

Observe that these operations are well defined for any two partially ordered multisets. Let us denote respectively by $\{\phi, \psi\}$ and $(\phi; \psi)$ the parallel and series composition of two partially ordered multisets of events ϕ and ψ.

[4] The partial orders $<$ and $\prec$ are viewed as subset of $X \times X$ and $Y \times Y$, and the maps f and g are viewed as subsets of $X \times E$ and $Y \times E$.

Proposition 9 *The minimal marking $M^{\{\phi_1,\phi_2\}}$ of $\{\phi_1,\phi_2\}$ is the sum of the minimal markings for ϕ_1 and ϕ_2 that is $M^{\phi_1} \otimes M^{\phi_2}$.*

PROOF : ${M^{\{\phi_1,\phi_2\}}}_A$ is defined by

$$ {M^{\{\phi_1,\phi_2\}}}_A = \max_{Y \in L(\{\phi_1,\phi_2\})} \mathsf{Pre}[\mathcal{F}(Y)]_A - \mathsf{Post}[Y]_A + \mathsf{Pre}[Y]_A $$

where $L(X)$ stands for the lower-closed subsets of X.

Observe that $Y \in L(\{\phi_1,\phi_2\})$ means that $Y = Y_1 \cup Y_2$ with $Y_1 \in L(\phi_1)$ and $Y_2 \in L(\phi_2)$, and that $\mathcal{F}(Y) = \mathcal{F}_{\phi_1}(Y_1) \cup \mathcal{F}_{\phi_2}(Y_2)$ — where $\mathcal{F}_\psi(Z)$ denotes the frontier *with respect to* ψ of the lower closed subset Z of ψ.

$$
\begin{array}{ll}
\multicolumn{2}{l}{{M^{\{\phi_1,\phi_2\}}}_A} \\
= & \max_{Y \in L(\{\phi,\psi\})} \mathsf{Pre}[\mathcal{F}(Y)]_A - \mathsf{Post}[Y]_A + \mathsf{Pre}[Y]_A \\[2ex]
= & \max_{Y_1 \in L(\phi_1), Y_2 \in L(\phi_2)} \left(\begin{array}{l} \mathsf{Pre}[\mathcal{F}_{\phi_1}(Y_1) \cup \mathcal{F}_{\phi_2}(Y_2)]_A \\ -\mathsf{Post}[Y_1 \cup Y_2]_A \\ +\mathsf{Pre}[Y_1 \cup Y_2]_A \end{array} \right) \\[2ex]
= & \max_{Y_1 \in L(\phi_1), Y_2 \in L(\phi_2)} \left(\begin{array}{l} \mathsf{Pre}[\mathcal{F}_{\phi_1}(Y_1)]_A - \mathsf{Post}[Y_1]_A + \mathsf{Pre}[Y_1]_A \\ +\mathsf{Pre}[\mathcal{F}_{\phi_2}(Y_2)]_A - \mathsf{Post}[Y_2]_A + \mathsf{Pre}[Y_2]_A \end{array} \right) \\[2ex]
= & \begin{array}{l} \left(\max_{Y_1 \in L(\phi_1)} \mathsf{Pre}[\mathcal{F}_{\phi_1}(Y_1)]_A - \mathsf{Post}[Y_1]_A + \mathsf{Pre}[Y_1]_A\right) \\ + \left(\max_{Y_2 \in L(\phi_2)} \mathsf{Pre}[\mathcal{F}_{\phi_2}(Y_2)]_A - \mathsf{Post}[Y_2]_A + \mathsf{Pre}[Y_2]_A\right) \end{array} \\[2ex]
= & {M^{\phi_1}}_A + {M^{\phi_2}}_A
\end{array}
$$

$\diamond$

Proposition 10 (step transitions and series or parallel composition)

1. *If $M_1 \xrightarrow{\phi_1} N_1$ and $M_2 \xrightarrow{\phi_2} N_2$ then $M_1 \otimes M_2 \xrightarrow{\{\phi_1,\phi_2\}} N_1 \otimes N_2$. In particular, letting ϕ_2 be the empty partially ordered multiset of events (so $M_2 = N_2$) which is the unit for parallel composition we have: if $M \xrightarrow{\phi} N$ then $M \otimes P \xrightarrow{\phi} N \otimes P$, for any marking P.*

2. *If* $M \xrightarrow{\{\phi_1,\phi_2\}} N$ *then there exists markings* M_1, M_2, N_1, N_2, P *such that:*
   (a) $M = M_1 \otimes M_2 \otimes P$
   (b) $N = N_1 \otimes N_2 \otimes P$
   (c) $M_1 \xrightarrow{\phi_1} N_1$
   (d) $M_2 \xrightarrow{\phi_2} N_2$
3. *If* $M \xrightarrow{(\phi_1;\phi_2)} N$ *then there exists a marking* Q *such that:*
   - $M \xrightarrow{\phi_1} Q$
   - $Q \xrightarrow{\phi_2} N$

PROOF : 1 By proposition 6 we have $M_i \sqsupseteq M^{\phi_i}$, and therefore $M_1 \otimes M_2 \sqsupseteq M^{\phi_1} \otimes M^{\phi_2}$. Because of the proposition 9 we have: $M_1 \otimes M_2 \sqsupseteq M^{\{\phi_1,\phi_2\}}$, and therefore there exists X such that $M_1 \otimes M_2 \xrightarrow{\{\phi_1,\phi_2\}} X$.

$$\begin{aligned} X &= M \otimes \mathsf{Post}[\phi] \oslash \mathsf{Pre}[\phi] \\ &= M_1 \otimes M_2 \otimes (\mathsf{Post}[\phi_1] \otimes \mathsf{Post}[\phi_2]) \oslash (\mathsf{Pre}[\phi_1] \otimes \mathsf{Pre}[\phi_2]) \\ &= M_1 \otimes \mathsf{Post}[\phi_1] \oslash \mathsf{Pre}[\phi_1] \otimes M_2 \otimes \mathsf{Post}[\phi_2] \oslash \mathsf{Pre}[\phi_2] \\ &= N_1 \otimes N_2 \end{aligned}$$

2 Let i be either 1 or 2. Because ϕ_i is a lower-closed subset of ϕ, one has $M \xrightarrow{\phi}$.

(a) Let $M_i = M_i^{\phi}$; because of proposition 6 which express the minimality of minimal markings, and of the proposition 9 which says that the minimal marking of a parallel composition is the sums of the minimal markings of its components, we know that $M \sqsupseteq M_1 \otimes M_2$, so there exists P such that $P = M \oslash (M_1 \otimes M_2)$ — so the results holds.

(c) and (d) Now let N_i be the final marking of the execution of ϕ_i from the initial marking M_i: we have $M_i \xrightarrow{\phi_i} N_i$, so (c) and (d) hold.

(b) From $M_1 \xrightarrow{\phi_1} N_1$ and $M_2 \xrightarrow{\phi_2} N_2$ by the previous item 1 we have: $M_1 \otimes M_2 \xrightarrow{\{\phi_1,\phi_2\}} N_1 \otimes N_2$. Still by the previous item 1 we deduce that: $M = M_1 \otimes M_2 \otimes P \xrightarrow{\{\phi_1,\phi_2\}} N_1 \otimes N_2 \otimes P$
Therefore by the unicity of the end marking, (b) holds.

3 The last point is an obvious computation.
Let $Q = M \otimes \mathsf{Post}[\phi_1] \oslash \mathsf{Pre}[\phi_1]$ — this expression is well defined because ϕ_1 is a lower closed subset of ϕ and $M \xrightarrow{\phi}$. Then one has $M \xrightarrow{\phi_1} Q$.
Now let Z be a lower closed subset of ϕ_2, and let $Z' = Z \cup \phi_1$; Z' is a lower closed subset of ϕ, and $\mathcal{F}_{\phi_2}(Z) = \mathcal{F}_\phi(Z')$. So one has

$$M \otimes \mathsf{Post}[Z] \otimes \mathsf{Post}[\phi_1] \oslash (\mathsf{Pre}[Z] \otimes \mathsf{Pre}[\phi_1]) \sqsupseteq \mathsf{Pre}[\mathcal{F}_\phi(Z')] = \mathsf{Pre}[\mathcal{F}_{\phi_2}(Z)]$$

that is:

$$Q \otimes \mathsf{Pre}[Z] \oslash \mathsf{Post}[Z] \sqsupseteq \mathsf{Pre}[\mathcal{F}_{\phi_2}(Z)]$$

$\diamond$

3 The PCLL calculus: sequent calculus

In this section we present the calculus PCLL. Actually our version slightly extends the published version, [14] but clearly was the author's project. Indeed, when his paper was printed he did not yet know the rules axiomatizing the inclusion of series-parallel partial orders [3] but this calculus was designed to incorporate such rules.

3.1 Formulae

Given a set of atomic formulae or propositional variables $\mathcal{P}$, that correspond to places from the Petri net viewpoint, formulae are defined by:

$$\mathcal{F} ::= \mathcal{P} \mid \mathbf{1} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \odot \mathcal{F} \mid \mathcal{F} \multimap \mathcal{F} \mid \mathcal{F} \setminus \mathcal{F} \mid \mathcal{F} \,/\, \mathcal{F}$$

So this calculus contains the following connectives:

- two multiplicative conjunctions:

- $\odot$ the non commutative conjunction of the Lambek calculus.
- $\otimes$ the commutative multiplicative conjunction of linear logic.

- the associated implications:
  - $\backslash$ associated with $\odot$ by: $A \odot (A \setminus B) \vdash B$.
  - $/$ associated with $\odot$ by $(B \mathbin{/} A) \odot A \vdash B$.
  - $\multimap$ associated with $\otimes$ by $A \otimes (A \multimap B) \vdash B$ and $(A \multimap B) \otimes A \vdash B$.

As we are going to see, either one can chose that the commutative conjunction entails the non commutative conjunction or the converse: of course this is determined by the structural rules managing the context. Here we use the version according to which $A \odot B \vdash A \otimes B$.

3.2 Contexts

Contexts, that are usually multisets or sequences of formulae, are here structured by series-parallel (partial) orders: they are partially ordered multisets of formulae (in the sense of section 2.2). The need for structured contexts is easily explained: the comma on the left-hand side of a sequent is an implicit conjunction. If we wish to have two kinds of conjunctions then we also need two kinds of "commas".

3.2.1 Reminder on series-parallel partial orders (SP-orders)

Series-parallel orders, SP-orders for short are the smallest class of finite strict partial orders containing all (empty) orders over a single point, and closed under series and parallel compositions already used in paragraph 2.2.4.

These binary operations are defined for any two orders ϕ and ψ with respective domains X and Y with $X \cap Y = \emptyset$, and both yield an order with domain $X \uplus Y$.

- *series composition or ordinal sum*

$$(\phi; \psi) = \phi \uplus \psi \uplus (X \times Y) \qquad \subseteq (X \uplus Y) \times (X \uplus Y)$$

- *parallel composition or disjoint sum*

$$\{\phi, \psi\} = \phi \uplus \psi \qquad \subseteq (X \uplus Y) \times (X \uplus Y)$$

The reader interested in SP-orders will find much more details in [19]. Here we just recall the basic properties that we need:

Proposition 11 *Two terms correspond to the same* SP*-order if and only if they are equal up to the algebraic properties of series and parallel compositions:*

$\{\ldots,\ldots\}$ *is associative and commutative*

$(\ldots;\ldots)$ *is associative*

Let us mention their famous characterization [29, 23]:

Proposition 12 *A finite order is* SP *if and only if its restriction to four points* a*,* b*,* c *and* d *never is* $\{(a,b),(c,b),(c,d)\}$.

which clearly entails:

Proposition 13 *Let* ϕ *be an* SP*-order of domain* X *and let* X' *be a subset of* X*. The restriction* $\phi' = \phi \cap (X' \times X')$ *is an* SP*-order as well. If* t *is an* SP*-term denoting* ϕ *one obtains a term denoting* ϕ' *from* t *by replacing each* $x \in (X - X')$ *by* ε *and reducing the term by applying the following equalities:* $(t;\varepsilon) = (\varepsilon;t) = \{t,\varepsilon\}$.[5]

Finally we have found in [3] a complete axiomatization for the inclusion of SP-orders as a rewriting system over SP-terms:

Proposition 14 *Let* ϕ *and* ϕ' *be two* SP*-orders with the same domain, and let* s *and* s' *two* SP*-terms denoting them. One has* $\phi \subseteq \phi'$ *if and only if* $s \longrightarrow^* s'$ *where* $\longrightarrow^*$ *is the reflexive and transitive closure of the following rewriting rules, where* $s[w]$

[5] Actually ε is the order on an empty domain, which is usually excluded from the class of SP-orders, although it is the unit for both series and parallel composition.

denotes an SP*-term containing an occurrence of the subterm* w*:*

SP-order inclusion

$$\begin{array}{rcl} s[\{(t;u),(t';u')\}] & \longrightarrow & s[(\{t,t'\};\{u,u'\})] \\ s[\{(t;u),u'\}] & \longrightarrow & s[(t;\{u,u'\})] \\ s[\{t,(t';u')\}] & \longrightarrow & s[(\{t,t'\};u')] \\ s[\{t,u\}] & \longrightarrow & s[(t;u)] \end{array}$$

SP-order equalities

$(\ldots;\ldots)$ associativity

$$\begin{array}{rcl} s[(t;(u;v))] & \longrightarrow & s[((t;u);v)] \\ s[((t;u);v)] & \longrightarrow & s[(t;(u;v))] \end{array}$$

$\{\ldots,\ldots\}$ associativity

$$\begin{array}{rcl} s[\{t,\{u,v\}\}] & \longrightarrow & s[\{\{t,u\},v\}] \\ s[\{\{t,u\},v\}] & \longrightarrow & s[\{t,\{u,v\}\}] \end{array}$$

$\{\ldots,\ldots\}$ commutativity

$$s[\{t,u\}] \longrightarrow s[\{u,t\}]$$

3.2.2 Contexts: SP-terms or SP-pomsets of formulae

Series composition, denoted by $(\ldots;\ldots)$ corresponds to the non commutative conjunction $\odot$, while parallel composition, denoted by $\{\ldots,\ldots\}$ corresponds to the commutative conjunction $\otimes$.

There are two ways to describe contexts depending on the precision we want, e.g. for proof search. Either they are viewed as SP-terms over formulae, or as SP-pomsets of formulae. The first description is better suited for writing down them and for implementing proof search, while the second is more abstract and consists in working with equivalence classes of terms.

Contexts as terms are defined as the following set of expressions in which $\mathcal{F}$ is the set of formulae:

$$\mathcal{C} ::= \mathcal{F} \mid \{\mathcal{C},\mathcal{C}\} \mid (\mathcal{C};\mathcal{C})$$

Contexts as SP*-orders* on multisets of formulae are described by the SP-terms denoting them which are the easiest way to write them down, but because of proposition 11 they are considered as equal exactly when they only differ up to the commutativity of $\{\ldots,\ldots\}$ and to the associativity of $\{\ldots,\ldots\}$ and $(\ldots;\ldots)$.

Denoting by $\uplus$ multiset union, the domain of a context is the multiset defined by:

$$|F| = \{F\} \quad |\{\Gamma, \Delta\}| = |(\Gamma; \Delta)| = |\Gamma| \uplus |\Delta|$$

The SP partial order Γ^{SP} associated with a context Γ is the subset of $|\Gamma| \times |\Gamma|$ defined by:

$$F^{\text{SP}} = \emptyset \quad \{\Gamma, \Delta\}^{\text{SP}} = \Gamma^{\text{SP}} \uplus \Delta^{\text{SP}} \quad (\Gamma; \Delta)^{\text{SP}} = \Gamma^{\text{SP}} \uplus \Delta^{\text{SP}} \uplus (|\Gamma| \times |\Delta|)$$

3.3 The rules of the calculus PCLL

3.3.1 Axioms

Axioms are identities:

$$\frac{}{A \vdash A}\, ax.$$

As usual, axioms can be limited to the case where A is a propositional variable.

3.3.2 Structural rules: which variant?

There are two variants of this calculus depending on whether we want the commutative conjunction to imply the non commutative one, or the converse, and both equally work from a formal point of view. This is set in the choice of the structural rules. Here, regarding that we have in mind executions of Petri nets, we chose to have the *augmenting context* variant which entails $A \odot B \vdash A \otimes B$. The corresponding structural rule is: [6]

$$\frac{\Gamma \vdash C}{\Gamma' \vdash C}\, aug(mentation) \quad \boxed{\text{if } |\Gamma| = |\Gamma'| \text{ and } \Gamma'^{\text{SP}} \supseteq \Gamma^{\text{SP}}}$$

This rule is not as non deterministic as it may seem. Indeed, because of the rewrite rules axiomatizing the inclusion of series-parallel orders given in proposition 14, $aug.$ is equivalent to the following rules on contexts as SP-terms, where

[6] The variant *diminishing context* which entails $A \otimes B \vdash A \odot B$ correspond to the structural rule:

$$\frac{\Gamma \vdash C}{\Gamma' \vdash C}\, entropy \quad \boxed{\text{if } |\Gamma| = |\Gamma'| \text{ and } \Gamma'^{\text{SP}} \subseteq \Gamma^{\text{SP}}}$$

$\Psi[\Phi]$ stands for a context containing Φ as a sub-context:

SP-equalities	SP-inclusions
$\dfrac{\Psi[((\Gamma;\Delta);\Theta)] \vdash C}{\Psi[(\Gamma;(\Delta;\Theta))] \vdash C}\,(asso;r)$	$\dfrac{\Psi[\{(\Gamma;\Gamma'),(\Delta;\Delta')\}] \vdash C}{\Psi[(\{\Gamma,\Delta\};\{\Gamma',\Delta'\})] \vdash C}\,(rew4)$
$\dfrac{\Psi[(\Gamma;(\Delta;\Theta))] \vdash C}{\Psi[((\Gamma;\Delta);\Theta)] \vdash C}\,(asso;l)$	$\dfrac{\Psi[\{\Gamma,(\Delta;\Delta')\}] \vdash C}{\Psi[(\{\Gamma,\Delta\};\Delta')] \vdash C}\,(rew3l)$
$\dfrac{\Psi[\{\{\Gamma,\Delta\},\Theta\}] \vdash C}{\Psi[\{\Gamma,\{\Delta,\Theta\}\}] \vdash C}\,(asso,r)$	$\dfrac{\Psi[\{(\Gamma;\Gamma'),\Delta\}] \vdash C}{\Psi[(\Gamma;\{\Delta,\Gamma'\})] \vdash C}\,(rew3r)$
$\dfrac{\Psi[\{\Gamma,\{\Delta,\Theta\}\}] \vdash C}{\Psi[\{\{\Gamma,\Delta\},\Theta\}] \vdash C}\,(asso,l)$	$\dfrac{\Psi[\{\Gamma,\Delta\}] \vdash C}{\Psi[(\Gamma;\Delta)] \vdash C}\,(rew2)$
$\dfrac{\Psi[\{\Gamma,\Delta\}] \vdash C}{\Psi[\{\Delta,\Gamma\}] \vdash C}\,(com,)$	

3.3.3 Other rules: connective introductions, and cut

Implication rules	
$\dfrac{\Gamma[B] \vdash C \quad \Delta \vdash A}{\Gamma[\{\Delta, A \multimap B\}] \vdash C}\multimap_h$	$\dfrac{\{A,\Gamma\} \vdash C}{\Gamma \vdash A \multimap C}\multimap_i$
$\dfrac{\Gamma[B] \vdash C \quad \Delta \vdash A}{\Gamma[(\Delta; A \setminus B)] \vdash C}\setminus_h$	$\dfrac{(A;\Gamma) \vdash C}{\Gamma \vdash A \setminus C}\setminus_i$
$\dfrac{\Gamma[B] \vdash C \quad \Delta \vdash A}{\Gamma[(B \,/\, A;\Delta)] \vdash C}/_h$	$\dfrac{(\Gamma;A) \vdash C}{\Gamma \vdash C \,/\, A}/_i$

Product/conjunction rules	
$\dfrac{\Gamma[(A;B)] \vdash C}{\Gamma[A \odot B] \vdash C}\,\odot_h$	$\dfrac{\Delta \vdash A \quad \Gamma \vdash B}{(\Delta;\Gamma) \vdash A \odot B}\,\odot_i$
$\dfrac{\Gamma[\{A,B\}] \vdash C}{\Gamma[A \otimes B] \vdash C}\,\otimes_h$	$\dfrac{\Delta \vdash A \quad \Gamma \vdash B}{\{\Delta,\Gamma\} \vdash A \otimes B}\,\otimes_i$

Unit rules	
$\dfrac{\Gamma \vdash C}{\{\Gamma, \mathbf{1}\} \vdash C}\,\mathbf{1}_h$	$\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}_i$

Cut rule
$\dfrac{\Gamma \vdash A \quad \Delta[A] \vdash B}{\Delta[\Gamma] \vdash B}\,cut$

3.3.4 Several remarks on PCLL calculus

Remark 15 (modus ponens) *The modus ponens provided by the calculus without the* $aug.$ *rule are:* $(B \,/\, A; A) \vdash B$, $(A; A \setminus B) \vdash B$ *and* $\{A, A \multimap B\} \vdash B$, *and* $\{A \multimap B, A\} \vdash B$. *But in this augmenting version of the calculus, where orders are allowed to augment, in addition to these expected modus ponens, we also have* $(A; A \multimap B) \vdash B$ *and* $(A \multimap B; A) \vdash B$ *— but neither* $\{A, A \setminus B\} \vdash B$ *nor* $\{A, B \,/\, A\} \vdash B$.

Remark 16 (1: unit for $\odot$ and $\otimes$) *The rules for* $\mathbf{1}$ *show that* $\mathbf{1}$ *is a unit for* $\odot$ *and* $\otimes$ *and the corresponding operations on contexts, respectively* $(\ldots;\ldots)$ *and* $\{\ldots,\ldots\}$. *In the rules* $\mathbf{1}_h$ *we could have decided to insert* $\mathbf{1}$ *anywhere in the context. However this alternative rule is not needed, since it is derivable using the rule which augments the context.*

Here is an obvious proposition which is useful to the main result:

Proposition 17 *Let* $M \in \Pi^{\otimes}$*; then* PCLL *proves* $\Gamma[M] \vdash C$ *if and only if* PCLL *proves* $\Gamma[\mathbf{1} \setminus M] \vdash C$

PROOF : Here are the proof trees:

$$\dfrac{\Gamma[M] \vdash C \qquad \dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}_i}{\Gamma[\mathbf{1} \setminus M] \vdash C}\,\setminus_h$$

$$\dfrac{\dfrac{\dfrac{\dfrac{M \vdash M}{\{\mathbf{1}, M\} \vdash M}\,\mathbf{1}_h}{(\mathbf{1}; M) \vdash M}\,aug.}{M \vdash \mathbf{1} \setminus M}\,\setminus_i \qquad \begin{matrix} \vdots\,\delta \\ \Gamma[\mathbf{1} \setminus M] \vdash C \end{matrix}}{\Gamma[M] \vdash C}\,cut$$

The second proof is not normal i.e. contains a cut, but when the proof δ is known it reduces to a cut-free proof. $\diamond$

The following essential property will be needed as well:

Theorem 18 (Cut-elimination and subformula property) *The cut-rule is redundant, and in a cut-free proof every formula of every sequent is a subformula of some formula of the conclusion sequent.*

PROOF : A semantical proof can be found in [7] and a syntactic one in appendix B. The proof is absolutely standard, the only novelty w.r.t. multiplicative linear logic or Lambek calculus being the commutation of cut and $aug.$. It results from the trivial monotonicity of order substitution w.r.t. inclusion: $\Delta' \subseteq \Delta \Rightarrow \Gamma[\Delta'] \subseteq \Gamma[\Delta]$ and $\Gamma'[X] \subseteq \Gamma[X] \Rightarrow \Gamma'[\Delta] \subseteq \Gamma[\Delta]$. $\diamond$

The property below allows us to freely denote a formula by one of its equivalent formulations:

Proposition 19 (Algebraic properties of the connectives) *If* $\Gamma \vdash C$ *is a provable sequent, and if one replaces each formula with an equivalent formula up to the commutativity and associativity of* $\otimes$ *and the associativity of* $\odot$*, one obtains again a provable sequent the proof of which is essentially similar.*

4 Encoding Petri nets

Although it is always easy to criticize previous work, let us nevertheless point out some drawbacks of previous coding of Petri nets into linear logic — for these previous codings, the reader is referred to the surveys [28, 8, 12]. There are objections both from the logic and concurrency viewpoints.

Events and initial state are encoded by *proper* axioms, which are logically not well behaved: cut-elimination and the subformula properties are not as pleasant as in a plain logical calculus — although the *standard* derivations of [28], which do correspond to the usual encoding of Petri nets in linear logic have such properties. Nevertheless still the coding suffers from the following mismatch: proper axioms (events) are reusable, so while Petri nets are a multiplicative phenomenon, we are not in the multiplicative calculus (I)MLL but in the multiplicative-exponential calculus (I)MELL, a system the decidability of which is yet unknown.[7]

What is more worrying from a concurrency viewpoint is the absence of the events from the sequent to be proved. Their occurrences during the firing is encoded by the proof, as well as their order of execution. The absence of some traces of events in the conclusion sequents prevents to study questions like the language of a net or net synthesis. Moreover even the sophisticated work of [9], also dealing with series-parallel executions via a subtle notion of normal proof, does not capture maximally concurrent execution as soon as parallelism is not only due to the events but also to the marking as shown in [15, 22], see appendix C.

Here we propose a coding which is inspired by the coding of context-free languages by Lambek grammars, see e.g. [16, 5, 24]. In this well-known approach, there is a lexicon mapping terminals to formulae, and the provability of a sequent $T_1, \ldots, T_n \vdash U$ in logical system (Lambek calculus) means that any sequences of terminals $a_1, \ldots, a_n$ whose respective types are $T_1, \ldots, T_n$ is produced by the non-terminal U. Our coding of Petri nets makes use of three kinds of formulae; we of course find again the notations introduced in the section 2.

Places Propositional variables, elements of $\mathcal{P}$.

[7] In any case, proper axioms are not closed under the substitution of a propositional variable with a formula, so there is no possibility to move to second order, if, for example, quantifi cation over places is needed for specifying the Petri-net behavior.

Markings Formulae with $\otimes$ as only connective. A marking with n_p tokens in each place p of a set Q of places, is denoted by the formula: $\bigotimes_{p \in Q} p^{n(p)}$ — regarding this part of the coding, there is no difference with previous work.

Events An event denoted by $\mathsf{Pre}[x] \setminus \mathsf{Post}[x]$ in section 2 is represented by this expression viewed as a formula of PCLL. Thus, the set of formulae denoting events is $\mathcal{E}^{\otimes \setminus \otimes} = \Pi^{\otimes} \setminus \Pi^{\otimes}$.

We can now state our main result precisely:

Theorem 20 *Given an* SP*-pomset ϕ of events (or of formulae in $\mathcal{E}^{\otimes \setminus \otimes}$) and two markings M and N (or formulae in $\Pi^{\otimes}$) the following propositions are equivalent:*

1. PCLL *proves* $(M; \phi) \vdash N$
2. $M \xrightarrow{\phi} N$

The $(2) \Rightarrow (1)$ is precisely proposition 23, which follows in section 6.

The $(1) \Rightarrow (2)$ results from a slightly more general result, proposition 27 of section 7, which concerns executive sequents — the ones that make sense w.r.t. to Petri nets:

Definition 21 *A sequent $\Gamma \vdash C$ is said to be* executive *whenever:*

- *all formulae of Γ are either markings ($\otimes$-only formulae) or events ($M \setminus N$ with M and N markings).*
- *C is a marking.*

The proposition 27 shows that whenever an executive sequent $\Gamma \vdash C$ is provable, one has $M \xrightarrow{\phi} C$, where

- ϕ is the restriction of the SP-order Γ to the events of Γ [8]
- M is the sum ($\otimes$) of all the markings in Γ.

One of the key points is that executive sequents are well behaved w.r.t. provability (proposition 24): normal proofs of executive sequents only contain executive sequents. The other is the substitution property on Petri net execution, proposition 7.

Before proving this, let us consider a small example.

[8] As seen in proposition 13 the restriction of an SP-order is also an SP-order.

5 A small example

Let us a consider a Petri net with two places a and b, and two events $x = a \setminus b$ and $y = b \setminus a$ — each of them moving one token from place to the other. If the initial marking is one token in each place, there are two different ways to execute x and y: either simultaneously and in this case the token used by y (resp. x) cannot be the one produced by x (resp. x), or sequentially and in this latter case, the token consumed by the second event can either be the one produced by the first event or the one that was already there.

Intuitively, how does our model take this difference into account? A given proof completely describes an execution: one can actually trace the consumption and production of tokens. Tokens are introduced by pairs in axioms, one being positive and the other negative — the usual notion of polarity, see any logic text book. They can be followed in each rule of a cut free proof, so in a sequent one can see by which events a token is produced and consumed. We indicate this by labeling the propositional variables with the number of the axiom they come from.

A proof net representation would provide a much clearer representation, since the tracing of the tokens corresponds to paths in proof nets. Unfortunately up to now there does not exists a proof net formalism for this calculus. Indeed, we could present proofs as proof structures of [1] but up to now there does not exists a sound and complete correctness criterion for recognizing proofs among them when the augmenting rule is allowed (their notion of proof net only allows for a $\otimes$ to be relaced by an $\odot$.)

Series composition corresponds to sequential composition of two executions, while parallel composition corresponds to the concurrent composition of two executions, so let us analyze the possible proofs of the sequents corresponding respectively to both ways to execute the Petri net:

$\underline{\{a, b\}; (a \setminus b; b \setminus a) \vdash a \otimes b}$ There are two essentially different proofs. In the first proof the second event consumes the token produced by the first event, and in the other proof, the second event consumes the token that was there since the initial marking.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{b_1 \vdash b_1 \quad a_2 \vdash a_2}{(b_1; b_1 \setminus a_2) \vdash a_2}\setminus_h \quad a_3 \vdash a_3}{((a_3; a_3 \setminus b_1); b_1 \setminus a_2) \vdash a_2}\setminus_h \quad b_4 \vdash b_4}{\{b_4, ((a_3; a_3 \setminus b_1); b_1 \setminus a_2)\} \vdash a_2 \otimes b_4}\otimes_i}{\{b_4, (a_3; (a_3 \setminus b_1; b_1 \setminus a_2))\} \vdash a_2 \otimes b_4}\,asso; r}{\{\{b_4, a_3\}, (a_3 \setminus b_1; b_1 \setminus a_2))\} \vdash a_2 \otimes b_4}\,aug.(rew3l)}{(\{a_3, b_4\}; (a_3 \setminus b_1; b_1 \setminus a_2)) \vdash a_2 \otimes b_4}\,aug.(rew2)}{(\{a_3 \otimes b_4\}; (a_3 \setminus b_1; b_1 \setminus a_2)) \vdash a_2 \otimes b_4}\otimes_h}{}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{b_1 \vdash b_1 \quad a_2 \vdash a_2}{(b_1; b_1 \setminus a_2) \vdash a_2}\setminus_h \quad \dfrac{a_3 \vdash a_3 \quad b_4 \vdash b_4}{(a_3; a_3 \setminus b_4) \vdash b_4}\setminus_h}{\{(b_1; b_1 \setminus a_2), (a_3; a_3 \setminus b_4)\} \vdash a_2 \otimes b_4}\otimes_i}{(\{a_3, b_1\}; \{a_3 \setminus b_4, b_1 \setminus a_2\}) \vdash a_2 \otimes b_4}\,aug.(rew4)}{(\{a_3, b_1\}; (a_3 \setminus b_4; b_1 \setminus a_2)) \vdash a_2 \otimes b_4}\,aug.(rew2)}{(a_3 \otimes b_1; (a_3 \setminus b_4; b_1 \setminus a_2)) \vdash a_2 \otimes b_4}\otimes_h
$$

$\underline{\{a, b\}; \{a \setminus b, b \setminus a\} \vdash a \otimes b}$ In this case there is a proof as well, similar to the second one above (skipping the final $aug.(rew2)$ rule); but there is no proof which would correspond to the first proof. It is indeed impossible (both intuitively and formally) that the token consumed by $b \setminus a$ is the one produced by $a \setminus b$, since these two events take place simultaneously. So it mandatory that there is no proof yielding $(*) \quad (\{a_3, b_4\}; \{a_3 \setminus b_1, b_1 \setminus a_2\}) \vdash a_2 \otimes b_4$ where the token b_1 is consumed and produced simultaneously — indeed simultaneously is even stronger than to fire events in both orders, as explained in proposition 4 and remark 5.

Let us explain this a bit. Every axiom introduces two occurrences of the same propositional variable, a positive one $\bar{p}$ and a negative one p, that can be traced in the proof down to the conclusion sequent $\Gamma \vdash C$. Token consumption corresponds to the conjunction of the two following prop-erties:

- The negative occurrence p occurs in a marking formula of Γ (token present at some moment) or in the N part of an event formula $M \setminus N$ (token produced by this event at some moment)
- The positive occurrence $\bar{p}$ occurs in the P part of an event $P \setminus Q$.

For our coding to make sense, it is mandatory that the logical system makes sure that the event consuming the token takes place once the token is present or has been produced – preventing, for instance, the existence of a proof yielding $(*)$ above, because $a_3 \setminus b_1 \not< b_1 \setminus a_2$ in $(\{a_3, b_4\}; \{a_3 \setminus b_1, b_1 \setminus a_2\})$. Fortunately, this is the case:

Proposition 22 *Let δ be a cut-free proof of an executive sequent $\Gamma \vdash C$ where the two occurrences p and $\bar{p}$ of the same propositional variables introduced by the same axiom of δ occur as follows:*

- *p in a marking formula $F[p] = N[p] \in \Pi^{\otimes}$ of Γ or in $N[p]$ the target part of an event formula $F[p] = M \setminus N[p] \in \mathcal{E}^{\otimes\setminus\otimes}$*
- *$\bar{p}$ in $P[\bar{p}] \setminus Q \in \mathcal{E}^{\otimes\setminus\otimes}$ of Γ*

We then have $F[p] < P[\bar{p}] \setminus Q$ in the SP*-order Γ.*

Intuitively, a token is present or already produced before it is consumed by another event.

PROOF : Using the fact that only executive sequents appear in a proof of an executive sequent (proposition 24) the proposition 25 will establish precisely this property. $\diamond$

6 From Petri nets to proofs in PCLL

Proposition 23 *Let ϕ be an* SP*-pomset of events of a Petri net, which can also be viewed as a context whose formulae are events.*

If $M \xrightarrow{\phi} N$ then PCLL *proves $(M; \phi) \vdash N$*

PROOF : We proceed by induction on the SP-term ϕ.

$\phi = P \setminus Q$ (ϕ is a single event) Since $M \xrightarrow{P\setminus Q} N$, there exist markings P, Q, R such that $M = P \otimes R$ and $N = Q \otimes R$.

$$\dfrac{\dfrac{\dfrac{\dfrac{P \vdash P \quad Q \vdash Q}{(P; P \setminus Q) \vdash Q} \setminus_h \quad R \vdash R}{\{(P; P \setminus Q), R\} \vdash Q \otimes R} \otimes_i}{(\{P, R\}; P \setminus Q) \vdash Q \otimes R} aug.}{(P \otimes R; P \setminus Q) \vdash Q \otimes R} \otimes_h$$

$\phi = \{\phi_1, \phi_2\}$ We know from 2 of proposition 10 that there exist markings M_1,M_2,N_1,N_2 and R such that:

- $M = M_1 \otimes M_2 \otimes R$
- $N = N_1 \otimes N_2 \otimes R$
- $M_1 \xrightarrow{\phi_1} N_1$
- $M_2 \xrightarrow{\phi_2} N_2$

By induction hypothesis we know that PCLL proves $(M_1; \phi_1) \vdash N_1$ and $(M_2; \phi_2) \vdash N_2$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(M_1; \phi_1) \vdash N_1 \quad (M_2; \phi_2) \vdash N_2}{\{(M_1; \phi_1), (M_2; \phi_2)\} \vdash N_1 \otimes N_2} \otimes_i}{(\{M_1, M_2\}; \{\phi_1, \phi_2\}) \vdash N_1 \otimes N_2} aug. \quad R \vdash R}{\{(\{M_1, M_2\}; \{\phi_1, \phi_2\}), R\} \vdash N_1 \otimes N_2 \otimes R} \otimes_i}{(\{M_1, M_2, R\}; \{\phi_1, \phi_2\}) \vdash N_1 \otimes N_2 \otimes R} aug.}{(\{M_1, M_2 \otimes R\}; \{\phi_1, \phi_2\}) \vdash N_1 \otimes N_2 \otimes R} \otimes_h}{(M_1 \otimes M_2 \otimes R; \{\phi_1, \phi_2\}) \vdash N_1 \otimes N_2 \otimes R} \otimes_h$$

$\phi = (\phi_1; \phi_2)$ We know from 3 of proposition 10 that there exists a marking Q such that $M \xrightarrow{\phi_1} Q$ and $Q \xrightarrow{\phi_2} N$ therefore, by induction hypothesis

PCLL proves $M; \phi_1 \vdash Q$ and $Q; \phi_2 \vdash N$.

$$\dfrac{\dfrac{(M;\phi_1) \vdash Q \quad (Q;\phi_2) \vdash N}{((M;\phi_1);\phi_2) \vdash N}\, cut}{(M;(\phi_1;\phi_2)) \vdash N}\text{ ";" associative}$$

This yields a non cut-free proof but when the whole proof is built, we can eliminate them, and thus obtain a cut-free proof. $\diamond$

7 From proofs in PCLL to Petri net execution

7.1 A property of the PCLL calculus on executive sequents

Recall from definition 21 that executive sequents of PCLL are the ones whose right-hand side is a marking, and whose left hand-side only consists in markings or events. Although it is more than a mere language restriction, executive sequents are closed under provability in PCLL calculus in the following sense:

Proposition 24 *Let δ be a cut free proof of an executive sequent; then each sequent in δ is itself an executive sequent. Consequently the only rules of δ are*

$$aug., \mathbf{1}_h, \backslash_h, \otimes_i, \otimes_h$$

and its axioms are either $\vdash \mathbf{1}$ *(*$\mathbf{1}_i$*) or* $M \vdash M$ *(*$ax.$*) with* $M \in \Pi^{\otimes}$.

PROOF : Because of the subformula property (proposition 18) only formulae of $\mathcal{E}^{\otimes\backslash\otimes} \uplus \Pi^{\otimes}$ can appear in δ. So δ does not contain any of the rules $\odot_h$, $\odot_i$, $/_h$, $/_i$, $\multimap_h$ or $\multimap_i$ since all these rules introduce a connective which does not appear in the conclusion sequent. To complete the proof, let us show that if the right-hand side of a sequent of δ contains a formula of $\mathcal{E}^{\otimes\backslash\otimes}$, then so does the right hand side of the sequent below it.

So assume there is a formula H containing the symbol $\backslash$ in the right-hand side of a sequent of δ — a cut free proof of an executive sequent $\Gamma \vdash C$. Now let us see that whatever the rule having this sequent as one of its premises we obtain an $\backslash$ symbol in the right-hand side of the conclusion sequent of the rule.

$\underline{aug., \mathbf{1}_h}$ Assume there is a formula of $\mathcal{E}^{\otimes\backslash\otimes}$ in the right hand-side of the premise of either of these two rules; then there is one in the right hand side of the conclusion sequent of either rule.

$\underline{\backslash_i}$ This rule would create in the right-hand side of the conclusion sequent a formula with two symbols $\backslash$'s, that is a formula which is not a subformula of the conclusion sequent of δ — this rule is not used below the problematic sequent.

$\underline{\backslash_h}$

- either H is kept in the right hand-side of the sequent, so there is a symbol $\backslash$ in the right hand-side of the conclusion sequent of the rule,
- or a formula $H \backslash X$ is created in the left-hand side of the conclusion sequent; this formula has at least two symbols $\backslash$, so it is a formula which is not a subformula of the conclusion sequent of δ, this subcase is impossible.

$\underline{\otimes_i}$ This would create a formula $H \otimes U$ which is not a subformula of the conclusion sequent of δ, this case is impossible.

$\underline{\otimes_h}$ H is kept in the right-hand side of the conclusion sequent.

The presence of a $\backslash$ symbol in the right hand-side of a sequent of δ would entail the presence of such a symbol in C, the right hand-side of the sequent $\Gamma \vdash C$ proved by δ, and this conflicts with $\Gamma \vdash C$ being an executive sequent.

As we have shown that δ only contains executive sequents, that is contains no $\backslash$ symbol in the right-hand side of any sequent, it is clear that the rule $\backslash_i$ is not used: indeed, it introduces a $\backslash$ symbol in the right-hand side of its conclusion sequent.

Since the proof only contains executive sequents the axioms can only be $M \vdash M$ with $M \in \Pi^{\otimes}$ or $\vdash \mathbf{1}$. $\diamond$

We can now come back to the properties discussed at the end of section 5 which lead us to proposition 22, that we can now prove formally. Given a provable executive sequent $\Gamma \vdash C$, remember from section 5 that the consumption of a token corresponds to the two occurrences of the same propositional variable introduced by the same axiom, such that:

- the negative occurrence p is either in a marking M of Γ (the token is present at some moment in the order) or in the par N of an event $M \setminus N$ (the token is produced by this event at some moment in the order)
- the positive occurrence $\bar{p}$ is in a formula P of an event $P \setminus Q$ of Γ

Although this is only needed to understand our our logical model represents the consumption of tokens, let us show the following property which shows that a token is only consumed after it is present or produced — the case $G[\bar{p}] = C$ corresponding to tokens which are present or produced at some moment but which are not consumed.

Proposition 25 *Let $\Gamma \vdash C$ be an executive sequent provable in* PCLL*, and let δ be a cut-free proof of $\Gamma \vdash C$ with only atomic axioms $p \vdash p$. Let p and $\bar{p}$ be the two occurrences of $\Gamma \vdash C$ coming from the same axiom $p \vdash \bar{p}$ — p is the negative occurrence of p in the sequent, and $\bar{p}$ the corresponding positive occurrence of the same propositional variable. Let $F[p]$ and $G[\bar{p}]$ be the formulae of $\Gamma \vdash C$ which respectively contain the occurrences p and $\bar{p}$. Then*

$F[p]$ *is a formula of Γ and if $F[p]$ is an event $F[p] = M \setminus N$ then the occurrence p is in N.*

$G[\bar{p}]$ *is either C or an event $G[\bar{p}] = P \setminus Q$ of Γ, and in this latter case, the occurrence $\bar{p}$ is in P and* we have $F[p] < G[\bar{p}]$ in the SP-order Γ.

PROOF : In a multiplicative calculus, no two propositional variables can be identified, so the we can easily trace propositional variables in a proof. In any rule, each occurrence of a propositional variable in the conclusion sequent corresponds to a single occurrence of the same variable in one of the premise sequent, and conversely each occurrence of a propositional variable in a premise sequent corresponds to one occurrence of the same propositional variable in the conclusion sequent.

Because of the polarity of p and $\bar{p}$, and because $C \in \Pi^{\otimes}$ and every formula of Γ is either in $\Pi^{\otimes}$ or $\Pi^{\otimes} \setminus \Pi^{\otimes}$ we necessarily have:

- $F[p]$ is a formula of Γ and if $F[p]$ is an event $F[p] = M \setminus N$ then the occurrence of p is in N

- $G[\bar{p}]$ is either C or an event $G[\bar{p}] = P \setminus Q$ of Γ, and in this latter case, the occurrence of $\bar{p}$ is in P

So the only thing left to be proved is $F[p] < G[\bar{p}]$ in the SP-order on Γ.

We proceed by induction on the cut-free proof, using proposition 24, noticing that the proof cannot be reduced to a single axiom $\mathbf{1}_i$.

ax The axiom is $p \vdash \bar{p}$ so $G[\bar{p}] = C$ and the result holds.

$aug.$ The formulae of the conclusion sequent and the premise sequent are the same. If $G[\bar{p}] = C$, then the result holds. If $G[\bar{p}] = P[\bar{p}] \setminus Q$, we already have $G[\bar{p}] < F[p]$ in the premise sequent, and, as the $aug.$ rule increases the order we also have $G[\bar{p}] < F[p]$ in the conclusion sequent.

$\mathbf{1}_h$ The formulae of the conclusion sequent and the premise sequent are the same, except the **1** in the conclusion sequent, which can neither be $F[p]$ nor $G[\bar{p}]$. If $G[\bar{p}] = C$, then the result holds. If $G[\bar{p}] = P[\bar{p}] \setminus Q$, we already have $G[\bar{p}] < F[p]$ in the premise sequent, and, as the the order among formulae different from the new occurrence of **1** is preserved under this rule, we also have $G[\bar{p}] < F[p]$ in the conclusion sequent.

$\otimes_i$ Let $\Delta \vdash A$ and $\Delta' \vdash B$ be the two premise sequents — hence $\Gamma = \{\Delta, \Delta'\}$ and $C = A \otimes B$. The two occurrences p and $\bar{p}$ belong to the same premise sequent. Assume this premise sequent is $\Delta \vdash A$, the other case being symmetrical. If in the premise sequent $\Delta \vdash A$ the occurence $\bar{p}$ is in A, then in the conclusion sequent $\Gamma \vdash C$ the occurrence $\bar{p}$ is in C ($G[\bar{p}] = A \otimes B = C$) and there is nothing to prove. If in the premise sequent $\Delta \vdash A$ the occurence $\bar{p}$ is not in A then it lies in the formula $G[\bar{p}]$ wich is not modified by this this rule, and the occurrence of p also is in the formula $F[p]$ which is also not modifed by this rule. By induction hypothesis we have $F[p] < G[\bar{p}]$ in Δ, hence we have $F[p] < G[\bar{p}]$ in $\{\Delta, \Delta'\} = \Gamma$.

$\otimes_h$ Let $\Delta[\{A, B\}] \vdash C$ be the premise sequent. Observe that the formula of this sequent which contains $\bar{p}$ can neither be A nor B, because fo the subformula property, so in the premise sequent $\bar{p}$ occurs in the same formula $G[\bar{p}]$ as in the conclusion sequent. If $G[\bar{p}]$ is C, then the results holds. Otherwise let us call $F'[p]$ the formula of $\Delta[\{A, B\}]$

wich contains the occurrence p — it can be A, B or any other formula of $\Delta[\{A, B\}]$ By induction hypothesis we have $F'[p] < G[\bar{p}]$ in $\Delta[\{A, B\}]$, and Γ is obtained from $\Delta[\{A, B\}]$ by identifying the two twin (w.r.t. the order) formulae A and B and calling $A \otimes B$ the result. If $F'[p] = A$ or $F'[p] = B$ then $F[p] = A \otimes B$ and we have $F[p] < G[\bar{p}]$ in $\Delta[A \otimes B] = \Gamma$. If $F'[p] \neq A$ or $F'[p] \neq B$ then $F[p] = F'[p]$ and we have $F[p] < G[\bar{p}]$ in $\Delta[A \otimes B] = \Gamma$.

$\backslash_h$ Let us recall the rule:

$$\cfrac{\cfrac{\Gamma[B] \vdash C \quad \Delta \vdash A}{\Gamma[(\Delta; A \setminus B)] \vdash C}\,\backslash_h}{\Theta \vdash C}\,=$$

Let $F'(p)$ and $G'(\bar{p})$ be the two formulae containing the occurrences p and $\bar{p}$ in the premisses sequents which necessarily are in the same premise sequent. There are several prossibilities:

<u>$F'[p]$ **in** $\Gamma[B]$ **and** $G'[\bar{p}] = C$</u> In this case $G'[\bar{p}] = G[\bar{p}] = C$, and the result holds.

<u>$F'[p]$ **and** $G'[\bar{p}]$ **in** $\Gamma[B]$</u> Observe that for polarity reasons $G'[\bar{p}] \neq B$. If $F[p] \neq B]$, by induction hypothesis, we have $F'[p] < G'[p]$ in $\Gamma[B]$ and $F'[p] = F[p]$, $G'[\bar{p}] = G[\bar{p}]$ and consequently $F[p] < G[\bar{p}]$ in Θ. If $F'[p] = B$ then $F[p] = A \setminus B$, and by induction hypothesis we have $F'[p] < B < G'[\bar{p}] = G[p]$ in $\Gamma[B]$. But then we have $F[p] = A \setminus B < G'[\bar{p}] = G[\bar{p}]$ in Θ.

<u>$F'[p]$ **and** $G'[\bar{p}]$ **in** Δ</u> By induction hypothesis, we have $F'[p] < G'[p]$ in Δ and $F'[p] = F[p]$, $G'[\bar{p}] = G[\bar{p}]$ and consequently $F[p] < G[\bar{p}]$ in Θ.

<u>$F'[p]$ **in** Δ **and** $G'[\bar{p}] = A$</u> In this case $F'[p] = F[p]$, $A \setminus B = G[\bar{p}]$ and we have $F[p] < A \setminus B = G[\bar{p}]$ in Θ.

$\diamond$

7.2 From PCLL proofs to concurrent executions

Let $\Gamma \vdash C$ be an executive sequent. We denote by $\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}$ the context obtained by replacing each marking M of Γ by the event $\mathbf{1} \setminus M$. Notice that from proposition 17 we know that PCLL proves $\Gamma \vdash C$ if and only if PCLL proves $\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}} \vdash C$.

Proposition 26 *If* PCLL *proves an executive sequent* $\Gamma \vdash C$ *then* $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}} C$.

PROOF : We proceed by induction on the height of a normal (cut-free) proof of $\Gamma \vdash C$. Because of 24, we know the only possible rules are $\setminus_h, \mathbf{1}_h, \otimes_h, \otimes_i, aug.$ and the only possible axioms are $\vdash \mathbf{1}$ $(\mathbf{1}_i)$ or $M \vdash M$ $(ax.)$ with $M \in \Pi^{\otimes}$.

<u>The last rule is an axiom</u> so it is either $\vdash \mathbf{1}$, or $M \vdash M$ with $M \in \Pi^{\otimes}$. Nothing to say: $\mathbf{1} \xrightarrow{\emptyset} \mathbf{1}$ and $\mathbf{1} \xrightarrow{\mathbf{1}\setminus M} M$ hold.

<u>The last rule is $\mathbf{1}_h$</u>

$$\frac{\Gamma \vdash C}{\{\Gamma, \mathbf{1}\} \vdash C}\ \mathbf{1}_h$$

By induction hypothesis we have $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}} C$, and $\mathbf{1} \xrightarrow{\mathbf{1}\setminus\mathbf{1}} \mathbf{1}$ by proposition 10.1 we have $\mathbf{1} = \mathbf{1} \otimes \mathbf{1} \xrightarrow{\{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}, \mathbf{1}\setminus\mathbf{1}\}} C \otimes \mathbf{1} = C$ and $\{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}, \mathbf{1} \setminus \mathbf{1}\} = \overline{\{\Gamma, \mathbf{1}\}}^{\mathbf{1}\setminus\mathcal{M}}$.

<u>The last rule is $aug.$</u>

$$\frac{\Gamma \vdash C}{\Gamma' \vdash C}\ aug(mentation) \quad \boxed{\text{if } |\Gamma| = |\Gamma'| \text{ and } \Gamma'^{\text{SP}} \supseteq \Gamma^{\text{SP}}}$$

By induction hypothesis we have $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}} C$ and since $\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}$ is a suborder of $\overline{\Gamma'}^{\mathbf{1}\setminus\mathcal{M}}$ we have $\mathbf{1} \xrightarrow{\overline{\Gamma'}^{\mathbf{1}\setminus\mathcal{M}}} C$ by proposition 4.

<u>The last rule is $\otimes_i$</u>

$$\frac{\Delta \vdash A \quad \Gamma \vdash B}{\{\Delta, \Gamma\} \vdash A \otimes B}\ \otimes_i$$

By induction hypothesis we have both $\mathbf{1} \xrightarrow{\overline{\Delta}^{\mathbf{1}\setminus\mathcal{M}}} A$ and $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}} B$. Thus by 1 of proposition 10 we have

$$\mathbf{1} = \mathbf{1} \otimes \mathbf{1} \xrightarrow{\overline{\{\Delta,\Gamma\}}^{\mathbf{1}\setminus\mathcal{M}} = \{\overline{\Delta}^{\mathbf{1}\setminus\mathcal{M}}, \overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}\}} A \otimes B$$

The last rule is $\otimes_h$

$$\frac{\Gamma[\{A, B\}] \vdash C}{\Gamma[A \otimes B] \vdash C} \otimes_h$$

By induction hypothesis we have: $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}[\{\mathbf{1}\setminus A, \mathbf{1}\setminus B\}]} C$.

Since $\mathbf{1} \xrightarrow{\{\mathbf{1}\setminus A, \mathbf{1}\setminus B\}} A \otimes B$ we can use the proposition 7.2 with $\psi = \{\mathbf{1} \setminus A, \mathbf{1} \setminus B\}$ and $x = \mathbf{1} \setminus A \otimes B$ — indeed $\mathbf{1}$ is the minimum marking of $\{\mathbf{1} \setminus A, \mathbf{1} \setminus B\}$. We thus obtain $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}[\mathbf{1}\setminus A\otimes B]} C$ that is: $\mathbf{1} \xrightarrow{\overline{\Gamma[A\otimes B]}^{\mathbf{1}\setminus\mathcal{M}}} C$

The last rule is $\setminus_h$

$$\frac{\Gamma[B] \vdash C \quad \Delta \vdash A}{\Gamma[(\Delta; A \setminus B)] \vdash C} \setminus_h$$

By induction hypothesis we have $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}[\mathbf{1}\setminus B]} C$ and we know that $\mathbf{1} \xrightarrow{\mathbf{1}\setminus A; A\setminus B} B$ holds.

So we can apply proposition 7.1 with $x = 1 \setminus B$ and $\psi = (\mathbf{1} \setminus A; A \setminus B)$. This yields: $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}[(\mathbf{1}\setminus A; A\setminus B)]} C$.

Since $\mathbf{1} \xrightarrow{\overline{\Delta}^{\mathbf{1}\setminus\mathcal{M}}} A$, we can again apply proposition 7.1 with $x = \mathbf{1} \setminus A$ and $\psi = \overline{\Delta}^{\mathbf{1}\setminus\mathcal{M}}$ We thus obtain $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}[(\overline{\Delta}^{\mathbf{1}\setminus\mathcal{M}}; A\setminus B)]} C$, that is: $\mathbf{1} \xrightarrow{\overline{\Gamma[(\Delta; A\setminus B)]}^{\mathbf{1}\setminus\mathcal{M}}} C$, ◇

Proposition 27 *If* PCLL *proves* $(M; \phi) \vdash C$ *where* ϕ *is an* SP*-pomset of events, then* $M \xrightarrow{\phi} C$.

PROOF : From proposition 26 we have $\mathbf{1} \xrightarrow{\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}}} C$. Because ϕ is an SP-pomset of events, $\overline{\Gamma}^{\mathbf{1}\setminus\mathcal{M}} = (\mathbf{1} \setminus M; \phi)$, and thus $\mathbf{1} \xrightarrow{\overline{(\mathbf{1}\setminus M;\phi)}^{\mathbf{1}\setminus\mathcal{M}}} C$. Applying proposition 8 with $\psi[\mathbf{1} \setminus M] = (\mathbf{1} \setminus M; \phi)$ we obtain $M \xrightarrow{\phi} C$. ◇

8 Future prospects

8.1 Petri net synthesis

The synthesis of Petri nets from formal languages is the following question:

> *Given a set of events A and a language $L \subseteq A^*$, does there exists a Petri net $\mathbf{R}$ with a marking M such that the sequences of events of the possible firings of $\mathbf{R}$ with M as its initial marking are precisely the words in L? If so, how can $\mathbf{R}$ be constructed from L?*

This question has been solved for deterministic context-free languages by Darondeau in [6]. Our encoding allows a logical formulation of this question — which has not yet been investigated, and possibly leads nowhere. Our logical approachcan be undertaken because the kind of logical systems we are using can be viewed as formal grammars, describing context-free languages.

Introduced by Lambek in his pioneering article [16] for natural language analysis via categorial grammars, Lambek-grammars reduce parsing to provability in a non commutative logic known as Lambek calculus — see also [5, 24]. This calculus is exactly the non commutative part of PCLL: connectives are restricted to $\backslash, /, \odot$ and context only allows $(\ldots;\ldots)$, and the only structural rule is associativity — contexts are sequences of formulae. A lexicon associates each terminal a in A with a finite sets of formulae $\mathcal{L}(a)$. A word $a_1 \ldots a_n$ of A^* belongs to the language generated by the Lambek grammars (i.e. the lexicon, they are lexicalized grammars) whenever for each i there exists a formula $t_i \in \mathcal{L}(a_i)$ such that the Lambek calculus proves

$$t_1, \ldots, t_n \vdash S$$

Lambek grammars describe all context-free languages [2, 5] (even if only $\backslash$ is allowed) [9] and only them [21, 5].

Assume the context-free language that we want to obtain as the language of a Petri net is defined by a Lambek grammar with only $\backslash$. We wish to obtain a Petri net whose sequences of events are the $a_1 \ldots a_n$'s such that $t_1;\ldots;t_n \vdash S$ with

[9] Given a context-free grammar G, put it into Greibach normal form G'. For each non terminal a, the lexicon $\mathcal{L}$ is defined by $X_1 \setminus X_2 \setminus \ldots X_n \setminus X \in \mathcal{L}(a)$ whenever G' contains the rule $X \to X_n \ldots X_1 a$. The Lambek grammar generates the same language as G' and so the same language as G.

$a_i \in \mathcal{L}(a_i)$. So the question is to find $M = M_1 \otimes \cdots \otimes M_m$ and to associate with each a_i one formula $\overline{a}_i$ of the shape $P_1 \otimes \cdots P_p \setminus Q_1 \otimes \cdots Q_q$ such that (1) and (2) are equivalent:

1. There exists N of the shape $N_1 \otimes N_2 \otimes \cdots \otimes N_n$ such that PCLL proves
$$M; \overline{a}_1; \overline{a}_2; \ldots ; \overline{a}_k \vdash N$$
2. for all i there exists $t_i \in \mathcal{L}(a_i)$ such that
$$t_1; \ldots ; t_n \vdash S$$

We are rather optimistic for this approach: indeed, there is a strong similarity between the terminal-type (the formula associated with a_i according to the lexicon) and the event-type (the formula corresponding to the event a_i).

$$\begin{array}{rcllll} t_i & = & X_i^1 \odot & \cdots & \odot X_i^{k_i} & \setminus \quad Y_i \\ \overline{a}_i & = & P_1 \otimes & \cdots & \otimes P_p & \setminus \quad Q_1 \otimes \cdots \otimes Q_q \end{array}$$

This suggests to use formula unification for solving net-synthesis questions.

8.2 Petri nets with credits

Our coding does not use the backward implication $/$. The meaning of such a connective is interesting from a computational viewpoint. Intuitively, an event $M \,/\, N$ consumes a marking that will appear later on: $(M \,/\, N; N) \vdash M$. This should correspond to the possibility to have a credit N that ought to be consumed later on.

This first application that come to the idea is to use this for protocols: one can specify that a token has to be received by an event.

The second one is computational linguistics, since the diminishing context version of this calculus restricted to first order formulae $M \setminus N \,/\, P$ is the one we used in [17] to describe minimalist grammars of Stabler [27] which describe mildly context-sensitive languages in a deductive framework — see [24] for a description of the general framework. Although the connection is yet unclear, our hope is to extract a Petri net model for parsing, as for instance pushdown automata correspond to context-free languages.

8.3 High order Petri nets

Our coding of Petri nets only makes use of first order implication. The PCLL calculus naturally enables the definition of high order Petri nets, where events could consume and produce markings or events (second order nets), or even higher order events. From the logical view point this is quite natural and should cause no trouble. For instance a higher order event $(R \otimes (N \setminus M)) \setminus ((P \setminus Q) \otimes S)$ consumes the marking R and the event $N \setminus M$ and produces a new event $P \setminus Q$ and a marking S. Clearly it is mandatory to bound the order (implication nesting) of formulae (e.g. to order at most 2, whever event can be consumed); indeed the whole PCLL logic leads to hardly interpretable or at least irrealistic mobile systems, too far away from actual computational processes. Most properties are preserved since the subformula property 18 guaranties that no formula of order more than p is needed for proving formulae of order p. So this approach suggests a neat treatment of mobile processes. This could also be combined with the notion of credit of the previous paragraph.

A Proof of the substitution property in concurrent executions (proposition 7)

In this section we prove the two parts of proposition 7. It should be observed that this proposition holds for any pomset of events, and not just for SP-pomsets of events.

Before proving it we need to know, how the lower closed subsets and their frontiers behave with respect to substitution.

A.1 Lower closed subsets, frontiers and substitution

Notation 28 *We are given a pomset ϕ with an occurrence of x, another pomset ψ and a lower closed subset Y of $\phi[x := \psi]$:*

- *We consider multisets as sets, that is we index the elements, and no two elements have the same index.*
- *Recall we only use $A - B$ when $A \supset B$.*
- *$C = A \uplus B$ means $C = A \cup B$ and $A \cap B = \emptyset$*
- *X the domain of ψ*
- *$\min(X)$ the set of the minimal elements of ψ.*
- *$\Phi = \phi[x := \psi]$.*
- *W the domain of Φ*
- *W' the domain of ϕ so $W' = (W - X) \uplus \{x\}$)*
- *$\mathbf{S}(x)$ is the set of all the immediate successors of x in ϕ, which is also the set of all the immediate successors of any of the maximal elements of ψ in Φ.*
- *$\mathbf{P}(x)$ is the set of all the immediate predecessors of x in ϕ, which is also the set of all the immediate predecessors of any of the minimal elements of ψ in Φ.*

Proposition 29 *Let Z be a lower closed subset of Φ, and let $X' = Z \cap X$. Then exactly one of the following cases holds:*

1. *$X' = X$*
   In this case $\mathcal{F}_\Phi(Z) = \mathcal{F}_\phi(Z')$ with $Z' = (Z - X) \uplus \{x\}$.

2. *$\emptyset \subsetneq X' \subsetneq X$*
   In this case $\mathcal{F}_\Phi(Z) = \mathcal{F}_\psi(X') \uplus (\mathcal{F}_\phi(Z') - \{x\})$ with $Z' = Z - X'$. Observe that $\mathcal{F}_\Phi(Z)$ can contain elements of $\min(X)$, when $\mathcal{F}_\psi(X')$ does.

3. *$X' = \emptyset$ and $Z \supset \mathbf{P}(x)$*
   In this case $\mathcal{F}_\Phi(Z) = (\mathcal{F}_\phi(Z) - \{x\}) \uplus \min(X)$

4. *$X' = \emptyset$ and $Z \not\supset \mathbf{P}(X)$*
   In this case $\mathcal{F}_\Phi(Z) = \mathcal{F}_\phi(Z)$

PROOF : The list is clearly an exhaustive one and all cases are disjoint. We have to check that the equalities for the frontiers hold.

1. $X' = X$ Let $Z' = (Z - X) \uplus \{x\}$. We have $\Phi|_{W-Z} = \phi|_{W'-Z'}$. Therefore $\mathcal{F}_\Phi(Z) = \mathcal{F}_\phi(Z')$.
2. $\emptyset \subsetneq X' \subsetneq X$ Let $Z' = Z - X'$.
   - *$\mathcal{F}_\phi(Z')$ is well defined.* We have to show that Z' is a lower closed subset of Φ without elements of X, hence a lower closed subset of ϕ. Observe that as soon as an element $z \notin X$ is above one element of X then it is above every element of X. Consequently if there would exists an $x \in X$ below an element z of Z' then all elements of X would be below z, and this would conflicts with $X' \neq X$ (that is $X \not\subset Z$).
   - $Z' \supseteq \mathbf{P}(X)$ Indeed Z is lower closed and contains an element of X while any element of $\mathbf{P}(X)$ is below any element of X .
   - $x \in \mathcal{F}_\phi(Z')$ Let z be an element such that $z < x$ in ϕ. Then there exists $p \in \mathbf{P}(X)$ such that $z \leq p$, hence $z \in Z'$, because Z' is lower closed and contains $\mathbf{P}(X)$ (previous item).

- *If* $z \in \mathcal{F}_\Phi(Z)$ *and* $\exists w \in X' \quad w \leq z$ *then* $z \in \mathcal{F}_\psi(X')$ Firstly let us show that $z \in X$. If $z \notin X$ since $\exists w \in X' \quad w \leq z$, we would have $z \geq u$ for all $u \in X$, conflicting with $X' \neq X$. Secondly, consider $z' \in X$ such that $z' < z$ w.r.t. ψ. Then we have $z' \in Z$, hence $z' \in X' = Z \cap X$. Consequently, $z \in \mathcal{F}_\psi(X')$.
- *If* $z \in \mathcal{F}_\Phi(Z)$ *and* $\neg\exists x \in X' \quad x \leq z$ *then* $z \in \mathcal{F}_\phi(Z') - \{x\}$ Observe that $z \neq x$ since $x \notin W$. Consider $u < z$ w.r.t. ϕ. As $u \neq x$ and $z \neq x$ this amounts to $u < z$ in Φ, hence $u \in Z$ and as $u \notin X$ we have $u \in Z'$. Hence $z \in \mathcal{F}_\phi(Z') - \{x\}$.
- *If* $z \in \mathcal{F}_\phi(Z') - \{x\}$ *then* $z \in \mathcal{F}_\Phi(Z)$ Notice that $z \in W - X$. Consider $u < z$ w.r.t. Φ. If $u \in X$ then for every element t of X we would have $t < z$ w.r.t. Φ, and $x < z$ w.r.t. ϕ; with $z \in \mathcal{F}_\phi(Z')$ this would entail $x \in Z'$, and this is impossible because $x \in \mathcal{F}_\phi(Z')$. Since $u \notin X$ and $z \neq x$ the relation $u < z$ w.r.t. Φ means $u < z$ w.r.t. ϕ, and therefore $u \in Z'$ hence $u \in Z$.
- *If* $z \in \mathcal{F}_\psi(X')$ *then* $z \in \mathcal{F}_\Phi(Z)$ Let $u < z$ w.r.t. Φ. If $u \in X$, then $u \in X' \subset Z$. If $u \notin X$, as $z \in X$, then $\exists p \in \mathbf{P}(X)\ u \leq p$. As $Z \supset \mathbf{P}(X)$ and Z is lower closed, $Z \ni u$.
- Consequently, $\mathcal{F}_\Phi(Z) = \mathcal{F}_\psi(X') \uplus (\mathcal{F}_\phi(Z') - \{x\})$.

3. $X' = \emptyset$ and $Z \supset \mathbf{P}(x)$

- $x \in \mathcal{F}_\phi(Z)$ Indeed $x \notin Z$ and $\forall u < x \quad \exists p \in \mathbf{P}(X) \quad u \leq p$, and as $p \in Z$ and Z is lower closed we have $u \in Z$.
- $\min(X) \subset \mathcal{F}_\Phi(Z)$ Indeed let $m \in \min(X)$ then $\forall u < m \exists p \in \mathbf{P}(X) \quad u \leq p$, and as $m \in Z$ and Z is lower closed we have $u \in Z$.
- *If* $z \notin X$ *and* $z \in \mathcal{F}_\Phi(Z)$ *then* $z \in \mathcal{F}_\phi(Z)$ Let $z \in \mathcal{F}_\Phi(Z)$, $z \notin X$, so $z \in W'$. Let $u < z$ w.r.t. ϕ. Then $u \in Z$, and therefore $u \neq x$. Hence $z \in \mathcal{F}_\phi(Z)$.
- *If* $z \neq x$ *and* $z \in \mathcal{F}_\phi(Z)$ *then* $z \in \mathcal{F}_\Phi(Z)$ Let $z \in \mathcal{F}_\phi(Z)$, $z \neq x$, so $z \in W$. Let $u < z$ w.r.t. Φ. Then $u \in Z$, and therefore $u \notin X$. Hence $z \in \mathcal{F}_\Phi(Z)$.
- Consequently $(\mathcal{F}_\Phi(Z) - \min(X)) = (\mathcal{F}_\phi(Z) - \{x\})$.

4. $X' = \emptyset$ and $Z \not\supset \mathbf{P}(X)$

- *If* $z \in \mathcal{F}_\Phi(Z)$ *then* $z \in \mathcal{F}_\phi(Z)$ We have $z \notin X$; otherwise as $\forall p \in \mathbf{P}(X) \quad p < z$, one would have $\mathbf{P}(X) \subset Z$. Therefore $z \in W'$. Let $u < z$ w.r.t. ϕ; then $u \neq x$ (otherwise $\mathbf{P}(X) \subset Z$). So $u \in W$ and $u \in Z$,hence $z \in \mathcal{F}_\phi(Z)$.
- *If* $z \in \mathcal{F}_\phi(Z)$ *then* $z \in \mathcal{F}_\Phi(Z)$ We have $z \neq x$; otherwise as $\forall p \in \mathbf{P}(X) \quad p < z$, one would have $\mathbf{P}(X) \subset Z$. Let $u < z$ w.r.t. Φ then $u \notin X$, $u \in W$, and thus $u < z$ w.r.t. ϕ; so $u \in Z$. Hence $z \in \mathcal{F}_\Phi(Z)$.

$\diamond$

A.2 Proof of proposition 7

To facilitate the computation we will extend the operation on markings to elements of the free abelian group over places and this corresponds to a negative number of tokens in a place. [10] It is harmless to compute markings using such expressions provided the result is a real a marking (all places have a positive or null exponent).

For simplification, we drop the $\otimes$ product. As we deal with element of the free abelian group over places, the expression $M \oslash N$ is always defined and means MN^{-1}.

The partial order $\sqsupseteq$ extends to elements of the free abelian group over places: $M \sqsupseteq N$ if for every place A the exponent of A in M is bigger than the exponent of A in N.

Observe that $\mathsf{Pre}[A \uplus B] = \mathsf{Pre}[A]\mathsf{Pre}[B]$ and $\mathsf{Post}[A \uplus B] = \mathsf{Post}[A]\mathsf{Post}[B]$ $(\diamond)$ and that $\mathsf{Pre}[A - B] = \mathsf{Pre}[A]\mathsf{Pre}[B]^{-1}$ and $\mathsf{Post}[A - B] = \mathsf{Post}[A]\mathsf{Post}[B]^{-1}$ since $A - B$ presupposes that $A \supseteq B$.

Proposition 30 (substitution property – 7.1 expansion) *Let*

- $\phi = (X, <, f)$ *be a partially ordered enumeration of events containing an occurrence* x *of* $P \setminus Q$ *(i.e.* $f(x) = P \setminus Q$*)*

[10] Linear logic notation, $\otimes$, oblige us to a “multiplicative” notation, while an additive one would be more intuitive. We would have vectors of integers, indicating how many tokens are present or missing in each place. It is nevertheless absolutely equivalent, since a free abelian group is the same as a $\mathbb{Z}$-module.

- $\psi = (Y, \prec, g)$ *be a partially ordered enumeration of events, such that* $Y \cap X = \emptyset$ *and such that* $P \xrightarrow{\psi} Q$

then one has:

$$(a) : \left(M_0 \xrightarrow{\phi}\right) \quad \Longrightarrow \quad \left(M_0 \xrightarrow{\phi[x:=\psi]}\right) \ : (b)$$

PROOF : Observe that $\mathsf{Post}[x] \ \ (\mathsf{Pre}[x])^{-1} = Q \ \ P^{-1} = \mathsf{Post}[X] \ \ \mathsf{Pre}[X]^{-1} \quad (\diamond)$.

We use the notation 28. Let Z be a lower closed subset of Φ, and let $X' = X \cap Z$, we have to show that $M_0 \ \ \mathsf{Post}[Z]\mathsf{Pre}[Z]^{-1} \sqsupseteq \mathcal{F}_\Phi(Z)$. We follow the cases of proposition 29.

1. $X' = X$
   In this case $\mathcal{F}_\Phi(Z) = \mathcal{F}_\phi(Z')$ with $Z' = (Z - X) \uplus \{x\}$.

   $$\mathsf{Pre}[Z] = \mathsf{Pre}[Z'] \ \ P \ \ \mathsf{Pre}[X]^{-1}$$

   $$\mathsf{Post}[Z] = \mathsf{Post}[Z'] \ \ Q \ \ \mathsf{Post}[X]^{-1}$$

   Therefore one has the following equalities where the last one is due to $(\diamond)$:

   $$\begin{array}{l} M_0 \ \ \mathsf{Post}[Z]\mathsf{Pre}[Z]^{-1} \\ = \ \ M_0 \ \ \mathsf{Post}[Z'] \ \ Q \ \ \mathsf{Post}[X]^{-1} \ \ (\mathsf{Pre}[Z'] \ \ P \ \ \mathsf{Pre}[X]^{-1})^{-1} \\ = \ \ M_0 \ \ \mathsf{Post}[Z'] \ \ \mathsf{Pre}[Z']^{-1} \ \ Q \ \ P^{-1} \ \ (\mathsf{Post}[X] \ \ \mathsf{Pre}[X]^{-1})^{-1} \\ = \ \ M_0 \ \ \mathsf{Post}[Z'] \ \ \mathsf{Pre}[Z']^{-1} \end{array}$$

   Because $M_0 \xrightarrow{\phi}$ and Z' is lower closed subset of ϕ, we know that

   $$M_0 \ \ \mathsf{Post}[Z']\mathsf{Pre}[Z']^{-1} \sqsupseteq \mathsf{Pre}[\mathcal{F}_\phi(Z')] = \mathsf{Pre}[\mathcal{F}_\Phi(Z)]$$

2. $\emptyset \subsetneq X' \subsetneq X$
   In this case $\mathcal{F}_\Phi(Z) = \mathcal{F}_\psi(X') \uplus (\mathcal{F}_\phi(Z') - \{x\})$ with $Z' = Z - X'$. Let us call $U = \mathcal{F}_\phi(Z') - \{x\}$.

   Because Z' is a lower closed subset of ϕ and $M_0 \xrightarrow{\phi}$ we have:

$$M_0\ \mathsf{Post}[Z']\ \mathsf{Pre}[Z']^{-1} \sqsupseteq \mathsf{Pre}[\mathcal{F}_\phi(Z')] = \mathsf{Pre}[U]\ \mathsf{Pre}[x] = \mathsf{Pre}[U]\ P \quad (*)$$

Because $P \xrightarrow{\psi}$ and X' is a lower closed subset of ψ we have:

$$P\ \ \mathsf{Post}[X']\ \ \mathsf{Pre}[X']^{-1} \sqsupseteq \mathsf{Pre}[\mathcal{F}_\psi(X')] \quad (**)$$

$$\begin{array}{ll} & M_0\ \ \mathsf{Post}[Z]\ \ \mathsf{Pre}[Z]^{-1} \\ = & M_0\ \ \mathsf{Post}[Z']\ \ \mathsf{Post}[X']\ \ \mathsf{Pre}[X']^{-1}\ \ \mathsf{Pre}[Z']^{-1} \\ \sqsupseteq & \mathsf{Pre}[U]\ \ P\ \ \mathsf{Post}[X']\ \ \mathsf{Pre}[X']^{-1} \qquad \text{because of } (*) \\ \sqsupseteq & \mathsf{Pre}[U]\ \ \mathsf{Pre}[\mathcal{F}_\psi(X')] \qquad \text{because of } (**) \\ = & \mathsf{Pre}[\mathcal{F}_\Phi(Z)] \end{array}$$

3. $X' = \emptyset$ and $Z \supset \mathbf{P}(x)$
   In this case $\mathcal{F}_\Phi(Z) = (\mathcal{F}_\phi(Z) - \{x\}) \uplus \min(X)$ Let U be the multiset of events such that $\mathcal{F}_\phi(Z) = \{x\} \uplus U$ and $\mathcal{F}_\Phi(Z) = U \uplus (\min(X))$.
   Because $M_0 \xrightarrow{\phi}$ and Z is lower closed subset of ϕ, we know that

   $$M_0\ \ \mathsf{Post}[Z]\ \ \mathsf{Pre}[Z]^{-1} \sqsupseteq \mathsf{Pre}[\mathcal{F}_\phi(Z)] = \mathsf{Pre}[x]\ \ \mathsf{Pre}[U]$$

   Because $P \xrightarrow{\psi}$, considering the $\emptyset$ which is lower closed and whose frontier in ψ is $\min(X)$ we have we have $P \sqsupseteq \mathsf{Pre}[\min(X)]$.
   Therefore

   $$M_0\ \mathsf{Post}[Z]\ \mathsf{Pre}[Z]^{-1} \sqsupseteq \mathsf{Pre}[x]\ \mathsf{Pre}[U] \sqsupseteq \mathsf{Pre}[\min(X)]\ \mathsf{Pre}[U] = \mathsf{Pre}[\mathcal{F}_\Phi(Z)]$$

4. $X' = \emptyset$ and $Z \not\supset \mathbf{P}(X)$
   In this case $\mathcal{F}_\Phi(Z) = \mathcal{F}_\phi(Z)$. Because $M_0 \xrightarrow{\phi}$ and Z is lower closed subset of ϕ, we know that

   $$M_0\ \ \mathsf{Post}[Z]\ \ \mathsf{Pre}[Z]^{-1} \sqsupseteq \mathsf{Pre}[\mathcal{F}_\phi(Z)] = \mathsf{Pre}[\mathcal{F}_\Phi(Z)]$$

$\diamond$

Proposition 31 (substitution property – 7.2 contraction) *Let*

- $\phi = (X, <, f)$ *be a partially ordered enumeration of events containing an occurrence* x *of* $P \setminus Q$ *(i.e.* $f(x) = P \setminus Q$*)*
- $\psi = (Y, \prec, g)$ *be a partially ordered enumeration of events, such that* $Y \cap X = \emptyset$ *and such that* $P \xrightarrow{\psi} Q$ with P being the minimum marking of ψ.

tehn one has:

$$(b) : \left(M_0 \xrightarrow{\phi[x:=\psi]}\right) \quad \Longrightarrow \quad \left(M_0 \xrightarrow{\phi}\right) : (a)$$

PROOF : We still use notation 28. Let Z' be a lower closed subset of ϕ. We have to show that $M_0 \;\; \mathsf{Pre}[Z']^{-1} \;\; \mathsf{Post}[Z'] \sqsupseteq \mathsf{Pre}[\mathcal{F}_\phi(Z')]$.

1. $x \notin \mathcal{F}_\phi(Z')$
   Let Z be the lower closed subset of Φ defined by $Z = Z'$ if $x \notin Z'$ and by $Z = (Z' - \{x\}) \uplus X$ if $x \in Z'$. Observe that $\mathsf{Post}[Z]\mathsf{Pre}[Z]^{-1} = \mathsf{Post}[Z']\mathsf{Pre}[Z']^{-1} \quad (*)$.
   We have $\mathcal{F}_\phi(Z') = \mathcal{F}_\Phi(Z)$. Because of $(*)$ we have:

   $$M_0 \;\; \mathsf{Post}[Z'] \;\; \mathsf{Pre}[Z']^{-1} = M_0 \;\; \mathsf{Post}[Z] \;\; \mathsf{Pre}[Z]^{-1}$$

   and since $M_0 \xrightarrow{\Phi}$ we have

   $$M_0 \;\; \mathsf{Post}[Z] \;\; \mathsf{Pre}[Z]^{-1} \sqsupseteq \mathsf{Pre}[\mathcal{F}_\Phi(Z)] = \mathsf{Pre}[\mathcal{F}_\phi(Z)]$$

2. $x \in \mathcal{F}_\phi(Z')$
   Let us call $U = \mathcal{F}_\phi(Z) - \{x\}$. Let X' be any lower closed subset of ψ. Then $Z = Z' \uplus X'$ is a lower closed subset of Φ and $\mathcal{F}_\Phi(Z) = U \uplus \mathcal{F}_\psi(X')$. Since $M_0 \xrightarrow{\Phi}$, we have:

   $$M_0 \;\; \mathsf{Post}[Z'] \;\; \mathsf{Post}[X'] \;\; \mathsf{Pre}[Z']^{-1} \;\; \mathsf{Pre}[X']^{-1} \sqsupseteq \mathsf{Pre}[U] \;\; \mathsf{Pre}[\mathcal{F}_\psi(X')]$$

   that is:

   $$(M_0 \; \mathsf{Pre}[U]^{-1} \; \mathsf{Post}[Z'] \; \mathsf{Pre}[Z']^{-1}) \; \mathsf{Post}[X'] \; \mathsf{Pre}[X']^{-1} \sqsupseteq \mathsf{Pre}[\mathcal{F}_\psi(X')]$$

Because this holds for any lower closed subset X' of ψ, and because P is the minimum marking of ψ, we have

$$(M_0 \quad \mathsf{Pre}[U]^{-1} \quad \mathsf{Post}[Z'] \quad \mathsf{Pre}[Z']^{-1}) \sqsupseteq P$$

and therefore

$$(M_0 \quad \mathsf{Post}[Z'] \;\; \mathsf{Pre}[Z']^{-1}) \sqsupseteq \mathsf{Pre}[U] \;\; P = \mathsf{Pre}[U]\mathsf{Pre}[x] = \mathsf{Pre}[\mathcal{F}_\phi(Z')]$$

$\diamond$

B Cut-elimination for the PCLL calculus

In the original paper [14], de Groote give a semantical proof of cut-elimination for a calculus which is the one we used, except that he only consider the following context rewrite rule :

$$\Gamma[\{A, B\}] \longrightarrow \Gamma[(A; B)]$$

Indeed the totality of rewriting rules for series parallel orders was not known by the moment he wrote the article.

So here we have to provide the reader with a proof of cut-elimination since our coding strongly relies on this result and its corollary: the subformula property. We give a syntactic proof of the result: it is lengthy, tedious and without surprise. The only point that needs to be looked at is whether $aug.$ commutes with cuts. The rest is given for sake of completeness.

We proceed by induction on $((d, p), h)$ where h is the height of the proof, d the maximal degree of a cut, and p the number of d-cuts. The degree of a formula is the height of the formula tree, and the degree of a cut is the degree of the cut-formula.

Proofs of height 1 are axioms, which are clearly cut-free.

If the last rule R of a proof is not a cut of maximal degree — R can be a cut of a lower degree — the transformation is rather obvious. The proof(s) obtained by suppressing R have a smaller height; by induction hypothesis, they can be turned into cut-free proofs; applying R to these cut-free proofs yields a proof with R as the only possible cut, and as R was not a maximal cut, the induction hypothesis also applies to this proof.

So we can assume that R is a cut of maximal degree.

$$\dfrac{\dfrac{\vdots\,\gamma}{\Gamma \vdash X}\,R^a \qquad \dfrac{\vdots\,\delta}{\Delta[X] \vdash C}\,R^f}{\Delta[\Gamma] \vdash C}\,\textit{cut } d$$

We are going to explore all possible values for R^a and R^f, and whatever these rules are, at least one of the following case transformation apply:

1. One of R^a or R^f is an axiom: the cut is suppressed.

2. R^a does not create the cut-formula, — so $R^a \neq \odot_i, \otimes_i, \otimes_i, \backslash_i, /_i, \multimap_i$ — in this case it is possible to apply R^a after the cut. So we apply the induction hypothesis to the proof minus R^a; its height is smaller, while the number of cut of degree d is strictly smaller or unchanged; so, by induction hyptohesis it can be turned into a cut-free proof. Reapplying R^a we obtain a proof which contains strictly less cut of degree d; hence, by induction hypothesis we are done.

3. If R^f does not create the cut formula, we proceed symmetrically.

4. If both R^a and R^f create the cut formula, then this cut of degree d is replaced with two cut of degree strictly smaller. Hence, we have less cut of degree d and by induction hypothesis we are done.

The main difference from related calculi (like MLL or the Lambek calculus) is the presence of the structural rules. When $R^a = aug.$ observe that $\Delta' \subset \Delta$ entails $\Gamma[\Delta'] \subset \Gamma[\Delta]$, and this allows to permute $aug.$ and R^a. Symmetrically, when R^f is $aug.$ observe that $\Gamma[X] \subset \Gamma'[X]$ entails $\Gamma[\Delta] \subset \Gamma'[\Delta]$: this allows to permute R^f and $aug.$

1 R^a **or** R^f **is an axiom** The final cut can be supressed.

2 R^a does not create X, the cut formula

R^a	Before reduction	After reduction
$aug.$	$\dfrac{\dfrac{\overset{\vdots\ \delta}{\Delta' \vdash X}}{\Delta \vdash X}\,aug.(\Delta' \subset \Delta) \qquad \overset{\vdots\ \gamma}{\Gamma[X] \vdash C}}{\Gamma[\Delta] \vdash C}\,cut\ d$	$\dfrac{\dfrac{\overset{\vdots\ \delta}{\Delta' \vdash X} \qquad \overset{\vdots\ \gamma}{\Gamma[X] \vdash C}}{\Gamma[\Delta'] \vdash C}\,cut\ d}{\Gamma[\Delta] \vdash X}\,aug.(\Gamma[\Delta'] \subset \Gamma[\Delta])$
$\odot_h$	$\dfrac{\dfrac{\overset{\vdots\ \gamma}{\Gamma[(A;B)] \vdash X}}{\Gamma[A \odot B] \vdash X}\,\odot_h \qquad \overset{\vdots\ \delta}{\Delta[X] \vdash C}}{\Delta[\Gamma[A \odot B]] \vdash C}\,cut\ d$	$\dfrac{\dfrac{\overset{\vdots\ \gamma}{\Gamma[(A;B)] \vdash X} \qquad \overset{\vdots\ \delta}{\Delta[X] \vdash C}}{\Delta[\Gamma[(A;B)]] \vdash C}\,cut\ d}{\Delta[\Gamma[A \odot B]] \vdash C}\,\odot_h$
$\otimes_h$	$\dfrac{\dfrac{\overset{\vdots\ \gamma}{\Gamma[\{A,B\}] \vdash X}}{\Gamma[A \otimes B] \vdash X}\,\otimes_h \qquad \overset{\vdots\ \delta}{\Delta[X] \vdash C}}{\Delta[\Gamma[A \otimes B]] \vdash C}\,cut\ d$	$\dfrac{\dfrac{\overset{\vdots\ \gamma}{\Gamma[\{A,B\}] \vdash X} \qquad \overset{\vdots\ \delta}{\Delta[X] \vdash C}}{\Delta[\Gamma[\{A,B\}]] \vdash C}\,cut\ d}{\Delta[\Gamma[A \otimes B]] \vdash C}\,\otimes_h$

$\backslash_h$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta[B]\vdash X} \quad \overset{\vdots\,\delta'}{\Delta'\vdash A}}{\Delta[(\Delta';A\backslash B)]\vdash X}\backslash_h \quad \overset{\vdots\,\gamma}{\Gamma[X]\vdash C}}{\Gamma[\Delta[(\Delta';A\backslash B)]]\vdash X}\,cut\ d$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta[B]\vdash X} \quad \overset{\vdots\,\gamma}{\Gamma[X]\vdash C}}{\Gamma[\Delta[B]]\vdash C}\,cut\ d \quad \overset{\vdots\,\delta'}{\Delta'\vdash A}}{\Gamma[\Delta[(\Delta';A\backslash B)]]\vdash X}\backslash_h$
$/_h$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta[B]\vdash X} \quad \overset{\vdots\,\delta'}{\Delta'\vdash A}}{\Delta[(B\,/\,A;\Delta')]\vdash X}/_h \quad \overset{\vdots\,\gamma}{\Gamma[X]\vdash C}}{\Gamma[\Delta[(B\,/\,A;\Delta')]]\vdash X}\,cut\ d$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta[B]\vdash X} \quad \overset{\vdots\,\gamma}{\Gamma[X]\vdash C}}{\Gamma[\Delta[B]]\vdash C}\,cut\ d \quad \overset{\vdots\,\delta'}{\Delta'\vdash A}}{\Gamma[\Delta[(B\,/\,A;\Delta')]]\vdash X}/_h$
$\multimap_h$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta[B]\vdash X} \quad \overset{\vdots\,\delta'}{\Delta'\vdash A}}{\Delta[\{\Delta',A\multimap B\}]\vdash X}\multimap_h \quad \overset{\vdots\,\gamma}{\Gamma[X]\vdash C}}{\Gamma[\Delta[\{\Delta',A\multimap B\}]]\vdash X}\,cut\ d$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta[B]\vdash X} \quad \overset{\vdots\,\gamma}{\Gamma[X]\vdash C}}{\Gamma[\Delta[B]]\vdash C}\,cut\ d \quad \overset{\vdots\,\delta'}{\Delta'\vdash A}}{\Gamma[\Delta[\{\Delta',A\multimap B\}]]\vdash X}\multimap_h$

3 R^f does not create X, the cut formula

R^f	Before reduction	After reduction
$aug.$	$\dfrac{\dfrac{}{\Delta \vdash X}\delta \quad \dfrac{\dfrac{}{\Gamma'[X] \vdash C}\gamma}{\Gamma[X] \vdash C}aug.(\Gamma'[X] \subset \Gamma[X])}{\Gamma[\Delta] \vdash C}cut\ d$	$\dfrac{\dfrac{\dfrac{}{\Delta \vdash X}\delta \quad \dfrac{}{\Gamma'[X] \vdash C}\gamma}{\Gamma'[\Delta] \vdash C}cut\ d}{\Gamma[\Delta] \vdash C}aug.(\Gamma'[\Delta] \subset \Gamma[\Delta])$
$\odot_h$	$\dfrac{\dfrac{}{\Delta \vdash X}\delta \quad \dfrac{\dfrac{}{\Gamma[(A;B)][X] \vdash C}\gamma}{\Gamma[A \odot B][X] \vdash C}\odot_h}{\Gamma[A \odot B][\Delta] \vdash C}cut\ d$	$\dfrac{\dfrac{\dfrac{}{\Delta \vdash X}\delta \quad \dfrac{}{\Gamma[(A;B)][X] \vdash C}\gamma}{\Gamma[(A;B)][\Delta] \vdash C}cut\ d}{\Gamma[A \odot B][\Delta] \vdash C}\odot_h$
$\otimes_h$	$\dfrac{\dfrac{}{\Delta \vdash X}\delta \quad \dfrac{\dfrac{}{\Gamma[\{A,B\}][X] \vdash C}\gamma}{\Gamma[A \otimes B][X] \vdash C}\otimes_h}{\Gamma[A \otimes B][\Delta] \vdash C}cut\ d$	$\dfrac{\dfrac{\dfrac{}{\Delta \vdash X}\delta \quad \dfrac{}{\Gamma[\{A,B\}][X] \vdash C}\gamma}{\Gamma[\{A,B\}][\Delta] \vdash C}cut\ d}{\Gamma[A \otimes B][\Delta] \vdash C}\otimes_h$

R^f	Before reduction	After reduction
$\setminus_h$	$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad \dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[B][X]\vdash C\end{array}\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash A\end{array}}{\Gamma[(\Gamma';A\setminus B)][X]\vdash C}\setminus_h}{\Gamma[(\Gamma';A\setminus B)][\Delta]\vdash C}\,\mathit{cut}\ d$	$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad\begin{array}{c}\vdots\,\gamma\\ \Gamma[B][X]\vdash C\end{array}}{\Gamma[B][\Delta]\vdash C}\,\mathit{cut}\ d\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash A\end{array}}{\Gamma[(\Gamma';A\setminus B)][\Delta]\vdash C}\setminus_h$
$/_h$	$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad \dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[B][X]\vdash C\end{array}\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash A\end{array}}{\Gamma[(B\,/\,A;\Gamma')][X]\vdash C}\,/_h}{\Gamma[(B\,/\,A;\Gamma')][\Delta]\vdash C}\,\mathit{cut}\ d$	$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad\begin{array}{c}\vdots\,\gamma\\ \Gamma[B][X]\vdash C\end{array}}{\Gamma[B][\Delta]\vdash C}\,\mathit{cut}\ d\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash A\end{array}}{\Gamma[(B\,/\,A;\Gamma')][\Delta]\vdash C}\,/_h$
$\multimap_h$	$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad \dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[B][X]\vdash C\end{array}\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash A\end{array}}{\Gamma[\{A\multimap B,\Gamma'\}][X]\vdash C}\multimap_h}{\Gamma[\{A\multimap B,\Gamma'\}][\Delta]\vdash C}\,\mathit{cut}\ d$	$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad\begin{array}{c}\vdots\,\gamma\\ \Gamma[B][X]\vdash C\end{array}}{\Gamma[B][\Delta]\vdash C}\,\mathit{cut}\ d\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash A\end{array}}{\Gamma[\{A\multimap B,\Gamma'\}][\Delta]\vdash C}\multimap_h$

R^f	Before reduction	After reduction
$\setminus_h$	$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta \vdash X\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[B]\vdash C\end{array} \quad \begin{array}{c}\vdots\,\gamma'\\ \Gamma'[X]\vdash A\end{array}}{\Gamma[(\Gamma'[X]; A \setminus B)] \vdash C}\,\setminus_h}{\Gamma[(\Gamma'[\Delta]; A \setminus B)] \vdash C}\,cut\ d$	$\dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[B]\vdash C\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array} \quad \begin{array}{c}\vdots\,\gamma'\\ \Gamma'[X]\vdash A\end{array}}{\Gamma'[\Delta] \vdash A}\,cut\ d}{\Gamma[(\Gamma'[\Delta]; A \setminus B)] \vdash C}\,\setminus_h$
$/_h$	$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta \vdash X\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[B]\vdash C\end{array} \quad \begin{array}{c}\vdots\,\gamma'\\ \Gamma'[X]\vdash A\end{array}}{\Gamma[(B / A; \Gamma'[X])] \vdash C}\,/_h}{\Gamma[(B / A; \Gamma'[\Delta])] \vdash C}\,cut\ d$	$\dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[B]\vdash C\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array} \quad \begin{array}{c}\vdots\,\gamma'\\ \Gamma'[X]\vdash A\end{array}}{\Gamma'[\Delta] \vdash A}\,cut\ d}{\Gamma[(B / A; \Gamma'[\Delta])] \vdash C}\,/_h$
$\multimap_h$	$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta \vdash X\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[B]\vdash C\end{array} \quad \begin{array}{c}\vdots\,\gamma\\ \Gamma'[X]\vdash A\end{array}}{\Gamma[\{\Gamma'[X], A \multimap B\}] \vdash C}\,\multimap_h}{\Gamma[\{\Gamma'[\Delta], A \multimap B\}] \vdash C}\,cut\ d$	$\dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[B]\vdash C\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array} \quad \begin{array}{c}\vdots\,\gamma'\\ \Gamma'[X]\vdash A\end{array}}{\Gamma'[\Delta] \vdash A}\,cut\ d}{\Gamma[\{\Gamma'[\Delta], A \multimap B\}] \vdash C}\,\multimap_h$

R^f	Before reduction	After reduction
$\otimes_i$	$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad \dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[X]\vdash A\end{array}\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash B\end{array}}{\{\Gamma[X],\Gamma'\}\vdash A\odot B}\otimes_i}{\{\Gamma[\Delta],\Gamma'\}\vdash A\odot B}\,\textit{cut } d$	$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad\begin{array}{c}\vdots\,\gamma\\ \Gamma[X]\vdash A\end{array}}{\Gamma[\Delta]\vdash A}\,\textit{cut } d\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash B\end{array}}{\{\Gamma[\Delta],\Gamma'\}\vdash A\odot B}\otimes_i$
$\odot_i$	$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad \dfrac{\begin{array}{c}\vdots\,\gamma\\ \Gamma[X]\vdash A\end{array}\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash B\end{array}}{(\Gamma[X];\Gamma')\vdash A\odot B}\odot_i}{(\Gamma[\Delta];\Gamma')\vdash A\odot B}\,\textit{cut } d$	$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta\\ \Delta\vdash X\end{array}\quad\begin{array}{c}\vdots\,\gamma\\ \Gamma[X]\vdash A\end{array}}{\Gamma[\Delta]\vdash A}\,\textit{cut } d\quad\begin{array}{c}\vdots\,\gamma'\\ \Gamma'\vdash B\end{array}}{(\Gamma[\Delta];\Gamma')\vdash A\odot B}\odot_i$

R^f	Before reduction	After reduction
$\setminus_i$	$\dfrac{\overset{\vdots\,\delta}{\Delta \vdash X} \quad \dfrac{\overset{\vdots\,\gamma}{(A;\Gamma[X]) \vdash B}}{\Gamma[X] \vdash A \setminus B}\setminus_i}{\Gamma[\Delta] \vdash A \setminus B}\mathit{cut}\ d$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta \vdash X} \quad \overset{\vdots\,\gamma}{(A;\Gamma[X]) \vdash B}}{(A;\Gamma[\Delta]) \vdash B}\mathit{cut}\ d}{\Gamma[\Delta] \vdash A \setminus B}\setminus_i$
$/_i$	$\dfrac{\overset{\vdots\,\delta}{\Delta \vdash X} \quad \dfrac{\overset{\vdots\,\gamma}{(\Gamma[X];A) \vdash B}}{\Gamma[X] \vdash B / A}/_i}{\Gamma[\Delta] \vdash B / A}\mathit{cut}\ d$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta \vdash X} \quad \overset{\vdots\,\gamma}{(\Gamma[X];A) \vdash B}}{(\Gamma[\Delta];A) \vdash B}\mathit{cut}\ d}{\Gamma[\Delta] \vdash B / A}/_i$
$\multimap_i$	$\dfrac{\overset{\vdots\,\delta}{\Delta \vdash X} \quad \dfrac{\overset{\vdots\,\gamma}{\{A,\Gamma[X]\} \vdash B}}{\Gamma[X] \vdash A \multimap B}\multimap_i}{\Gamma[\Delta] \vdash A \multimap B}\mathit{cut}\ d$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta \vdash X} \quad \overset{\vdots\,\gamma}{\{A,\Gamma[X]\} \vdash B}}{\{A,\Gamma[\Delta]\} \vdash B}\mathit{cut}\ d}{\Gamma[\Delta] \vdash A \setminus B}\multimap_i$

4 Both R^a and R^f create the cut-formula

	Before reduction	After reduction
$\odot$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta \vdash U} \quad \overset{\vdots\,\delta'}{\Delta' \vdash V}}{(\Delta;\Delta') \vdash U \odot V}\odot_i \quad \dfrac{\overset{\vdots\,\gamma}{\Gamma[(U;V)] \vdash C}}{\Gamma[U \odot V] \vdash C}\odot_h}{\Gamma[(\Delta;\Delta')] \vdash C}\,\mathit{cut}\ d$	$\dfrac{\overset{\vdots\,\delta}{\Delta \vdash U} \quad \dfrac{\overset{\vdots\,\delta'}{\Delta' \vdash V} \quad \overset{\vdots\,\gamma}{\Gamma[(U;V)] \vdash C}}{\Gamma[(U;\Delta')] \vdash C}\,\mathit{cut} < d}{\Gamma[(\Delta;\Delta')] \vdash C}\,\mathit{cut} < d$
$\otimes$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Delta \vdash U} \quad \overset{\vdots\,\delta'}{\Delta' \vdash V}}{\{\Delta, \Delta'\} \vdash U \otimes V}\otimes_i \quad \dfrac{\overset{\vdots\,\gamma}{\Gamma[\{U,V\}] \vdash C}}{\Gamma[U \otimes V] \vdash C}\otimes_h}{\Gamma[\{\Delta,\Delta'\}] \vdash C}\,\mathit{cut}\ d$	$\dfrac{\overset{\vdots\,\delta}{\Delta \vdash U} \quad \dfrac{\overset{\vdots\,\delta'}{\Delta' \vdash V} \quad \overset{\vdots\,\gamma}{\Gamma[\{U,V\}] \vdash C}}{\Gamma[\{U,\Delta'\}] \vdash C}\,\mathit{cut} < d}{\Gamma[\{\Delta,\Delta'\}] \vdash C}\,\mathit{cut} < d$

	Before reduction	After reduction
$\setminus$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{(U;\Delta)\vdash V}}{\Delta\vdash U\setminus V}\setminus_i \quad \dfrac{\overset{\vdots\,\gamma}{\Gamma[V]\vdash C}\quad \overset{\vdots\,\gamma'}{\Gamma'\vdash U}}{\Gamma[(\Gamma';U\setminus V)]\vdash C}\setminus_h}{\Gamma[(\Gamma';\Delta)]\vdash C}\,cut\ d$	$\dfrac{\dfrac{\overset{\vdots\,\gamma'}{\Gamma'\vdash U}\quad \overset{\vdots\,\delta}{(U;\Delta)\vdash V}}{(\Gamma';\Delta)\vdash V}\,cut<d \quad \overset{\vdots\,\gamma}{\Gamma[V]\vdash C}}{\Gamma[(\Gamma';\Delta)]\vdash C}\,cut<d$
$/$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{(\Delta;U)\vdash V}}{\Delta\vdash V\,/\,U}/_i \quad \dfrac{\overset{\vdots\,\gamma}{\Gamma[V]\vdash C}\quad \overset{\vdots\,\gamma'}{\Gamma'\vdash U}}{\Gamma[(V\,/\,U;\Gamma')]\vdash C}/_h}{\Gamma[(\Delta;\Gamma')]\vdash C}\,cut\ d$	$\dfrac{\dfrac{\overset{\vdots\,\gamma'}{\Gamma'\vdash U}\quad \overset{\vdots\,\delta}{(\Delta;U)\vdash V}}{(\Delta;\Gamma')\vdash V}\,cut<d \quad \overset{\vdots\,\gamma}{\Gamma[V]\vdash C}}{\Gamma[(\Delta;\Gamma')]\vdash C}\,cut<d$
$\multimap$	$\dfrac{\dfrac{\overset{\vdots\,\delta}{\{U,\Delta\}\vdash V}}{\Delta\vdash U\multimap V}\multimap_i \quad \dfrac{\overset{\vdots\,\gamma}{\Gamma[V]\vdash C}\quad \overset{\vdots\,\gamma'}{\Gamma'\vdash U}}{\Gamma[\{\Gamma',U\multimap V\}]\vdash C}\multimap_h}{\Gamma[\{\Gamma',\Delta\}]\vdash C}\,cut\ d$	$\dfrac{\dfrac{\overset{\vdots\,\gamma'}{\Gamma'\vdash U}\quad \overset{\vdots\,\delta}{\{U,\Delta\}\vdash V}}{\{\Gamma',\Delta\}\vdash V}\,cut<d \quad \overset{\vdots\,\gamma}{\Gamma[V]\vdash C}}{\Gamma[\{\Gamma',\Delta\}]\vdash C}\,cut<d$

C A remark on previous codings

In his thesis and paper [10, 9], Gehlot offers an encoding of Petri nets in Multiplicative Linear Logic (MLL, that is the commutative part of PCLL) and a technique of proof reduction which is a candidate for obtaining a maximally concurrent firing of a Petri net, as said in [10]:

> *Following these intuitions, it is desirable to provide a set of rewrite rules which will take proofs such as 1 and 2 and convert them to a "maximally concurrent" proof such as 3. This process resembles the cut elimination results from proof theory, but must differ in some ways since the cut elimination is being carried out in a theory in which cut elimination is impossible. [. . .] For the purposes of the current paper we offer a set of rewrite rules below which work for interesting cases we have studied, including the example of this section.*

Let us briefly describe his encoding:

- the markings are encoded as we do here by formulae of $\Pi^{\otimes}$,
- an event x corresponds to a proper axiom $\mathsf{Pre}[x] \vdash \mathsf{Post}[x]$ with $\mathsf{Pre}[x], \mathsf{Post}[x] \in \Pi^{\otimes}$
- as there are only formulae in $\Pi^{\otimes}$ the proofs of MLL can be made with only two rules, which are seq (sequential composition,cut) and $sync$ ($\otimes_i$ followed by $\otimes_h$):

$$\frac{M \vdash N \quad M' \vdash N'}{M \otimes M' \vdash N \otimes N'}\, sync$$

- the execution is described by the proof: $sync$ corresponds to the parallel composition and seq to sequential composition.

His notion of proof reduction generalises cut-elimination for a system with proper axioms: it also includes rule permutations. These reduction rules are aimed at reaching a maximally concurrent firing (among series-parallel executions, since the model does not depict others).

Unfortunately, in his thesis Künzle [15] has given a counter example to this claim, see also [22]

Consider the Petri net, with three places a, b, c and two events $x : (a \vdash b)$ and $y : (b \vdash c)$ as proper axioms. Assume there is one token in a and one in b. The following proof, corresponding to firing x then y (i.e. the SP-order) $(x; y)$ is normal according to Gehlot's definition:

$$\dfrac{\dfrac{a \vdash b \quad b \vdash c}{a \vdash c}\, seq \qquad b \vdash b}{a \otimes b \vdash b \otimes c}\, sync$$

Nevertheless it is not a maximally concurrent firing, even in the restricted classes of series-parallel executions. Indeed, the following proof shows that firing simultaneously x and y (i.e. the SP-order $\{x, y\}$) is possible as well and more parallel:

$$\dfrac{a \vdash b \quad b \vdash c}{a \otimes b \vdash b \otimes c}\, sync$$

The reason why is the presence of tokens in the initial marking, that can be added with $sync$. This kind of parallelism due to the initial marking is not taken into account by the proof reduction of Gehlot which only handles structural parallelism, i.e. the parallelism which only depends on the event structure of the Petri net.

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